Coupling two unimodal maps with simple kneading sequences

Bastien Fernandez¹ and Miaohua Jiang²

August 8, 2002

¹Centre de Physique Théorique (FRUMAM)
CNRS Luminy Case 907
13288 Marseille CEDEX 09 France
bastien@cpt.univ-mrs.fr
²Department of Mathematics
Wake Forest University
Winston Salem NC 27109 USA
jiangm@wfu.edu

Abstract

The dynamics of a system of two diffusively coupled identical unimodal maps is investigated. The unimodal maps have simple kneading sequences which imply that their orbits are all asymptotically periodic with period a power of two. Analogous results are obtained for the coupled system of all coupling parameters. We prove eventual periodicity of the position with respect to the diagonal of the square phase space and asymptotic periodicity for orbits whose coordinates have the same sign. Finally, we give a global condition for the existence of symmetric orbits that are examples of orbits satisfying the previous property.

AMSCSN: 37C05, 37E99.

1 Definitions and main results

1.1 Introduction

Coupled map lattices (CML) are models for the time evolution of reaction-diffusion systems. They consist of discrete time lattice dynamical systems with
nonlinear forcing and diffusive coupling [3, 9]. Mathematical studies of CML have mainly considered either weakly coupled or strongly coupled systems. In the weakly coupled case, small perturbations of uncoupled (infinite) lattices of hyperbolic maps are analyzed using techniques from ergodic theory and statistical mechanics [2, 7, 8, 11]. In strongly coupled finite lattices, various regimes of synchronization take place depending on the normal Lyapunov spectrum [1, 12].

In this paper, we consider a coupled map lattice of two sites with unimodal local maps. We describe its global asymptotic dynamics, for all coupling parameters, independent of the explicit algebraic expression of the local map.

We need to recall definitions of a unimodal map and an S-unimodal map, see e.g. [4].

A unimodal map is a map $f$ on $[-1, 1]$ satisfying the following properties.

1. $f(0) = 1$;
2. $f$ is strictly increasing on $[-1, 0]$ and strictly decreasing on $[0, 1]$;
3. $f$ is $C^1$ and $f'(x) = 0$ if and only if $x = 0$;
4. $f([f(1), 1]) \subset [f(1), 1]$.

An S-unimodal map is a $C^3$ unimodal map with negative Schwartzian derivative [4]. For instance, the quadratic map $f_a(x) = 1 - ax^2$ with $0 < a \leq 2$ is an S-unimodal map.

Given a unimodal map $f$, the coupled map lattice of two sites we study in this paper is the dynamical system $(\mathcal{M}, F_c)$ where the phase space is the square $\mathcal{M} = [f(1), 1]^2$ and where the map $F_c$ is defined as follows.

$$F_c(x, y) = \left((1 - \epsilon)f(x) + \epsilon f(y), (1 - \epsilon)f(y) + \epsilon f(x)\right), \quad (x, y) \in \mathcal{M},$$

where $\epsilon \in [0, \frac{1}{2}]$ is the coupling parameter. The map $F_c$ can be viewed as a composition of two maps: $F_c = L_c \circ F_0$, where the coupling operator

$$L_c(x, y) = ((1 - \epsilon)x + \epsilon y, (1 - \epsilon)y + \epsilon x)$$

is a linear convex combination of coordinates. It follows from the property (4) of $f$ that $F_c$ maps $\mathcal{M}$ into itself. Thus, the dynamical system is well-defined. The orbit with an initial point $(x, y)$ is denoted by the sequence $\{(x^t, y^t)\}_{t \in \mathbb{N}}$ with $(x^0, y^0) = (x, y)$ and $(x^{t+1}, y^{t+1}) = F_c(x^t, y^t)$ for all $t \in \mathbb{N}$. For convenience, our index set $\mathbb{N}$ starts from 0.

The map $F_c$ has a symmetry. It commutes with the map $\theta: (x, y) \mapsto (y, x)$. An orbit with $x^t \neq y^t$, $t \in \mathbb{N}$ is said to be symmetric if there exists $t_0 \in \mathbb{N}$ such that $(x^{t_0}, y^{t_0}) = \theta(x^0, y^0)$. We take $t_0$ to be the smallest of such integers. By
symmetry, every symmetric orbit is periodic with a period $2t_0$. This symmetry implies that the diagonal of $\mathcal{M}$, $\{(x, x) : x \in [f(1), 1]\}$, is invariant under $F_\epsilon$. Any orbit of $F_\epsilon$ lying in the diagonal is said to be homogeneous. Any orbit satisfying the condition $\lim_{t \to \infty} x^t - y^t = 0$ is said to be asymptotically homogeneous.

With respect to the coupling parameter $\epsilon$, the map $F_\epsilon$ has another symmetry $F_{1-\epsilon}(x, y) = \theta(F_\epsilon(x, y))$ as observed in [6]. Therefore, when $\epsilon \in [\frac{1}{2}, 1]$, the orbit of $F_\epsilon$ starting from $(x, y)$ is the sequence $\{\theta^t(x', y')\}$ where $\{(x', y')\}$ is the orbit of $F_{1-\epsilon}$ with $1 - \epsilon \in [0, \frac{1}{2}]$ starting from the same initial point. Hence, all our conclusions extend to the dynamics of $F_\epsilon$ when $\epsilon \in [\frac{1}{2}, 1]$.

Both analytic and numerical studies of systems of two coupled quadratic maps or systems of two coupled logistic maps can be found in the literature (the logistic map is the map $x \mapsto \mu x(1-x)$). These studies determined the parameter regions for the existence of some periodic orbits, investigated their bifurcations [5, 9, 10], and described the structure of attractors [6]. Even though the local map $f$ has relatively simple dynamics, these studies have demonstrated that such systems possess rich and interesting dynamics such as multistability, i.e., the co-existence of a large number (as large as one wants) of distinct stable periodic orbits. Moreover, the symmetric periodic orbits were shown to go through a Hopf bifurcation to create an invariant closed curve [5, 10].

On the other hand, independent of the explicit algebraic expression of $f$, two simple properties can be easily observed for $F_\epsilon$.
1. Every pair of hyperbolic periodic orbits of $f$ is a hyperbolic periodic orbit of $F_\epsilon$. Thus, it can be continued to obtain a periodic orbit for $F_\epsilon$ when $\epsilon$ is sufficiently small.
2. Let $K = \max_{x \in [f(1), 1]} |f'(x)|$, the condition $K(1-2\epsilon) < 1$ (which holds when $\epsilon$ is sufficiently close to $1/2$) implies that every orbit is asymptotically homogeneous, a property also known as synchronization. This is an immediate consequence of the relation

$$x^{t+1} - y^{t+1} = (1-2\epsilon)(f(x^t) - f(y^t)),$$

which directly follows from the definition of $F_\epsilon$.

To learn more about the asymptotic behavior of $(\mathcal{M}, F_\epsilon)$, especially for intermediate values of $\epsilon$, we consider unimodal maps $f$ with kneading sequences which imply that every orbit of $f$ is asymptotically periodic with period a power of 2 (see Proposition 1.1). Using properties of these sequences, we obtain analogous results for the coupled map lattice, namely, eventual periodicity of the position with respect to the diagonal (Theorem 1.2) and asymptotic periodicity of orbits with coordinates having the same sign (Theorem 1.3). Moreover,
we give a global condition for the existence of symmetric orbits. These orbits are examples of previous conclusions. They are believed to be the first non-homogeneous orbits to emerge when \( \epsilon \) is relaxed from the synchronization region. To prove these results, we first review dynamical properties of unimodal maps (with prescribed kneading sequences) in Section 2. Then, a precise description of the iterates of any point under the coupled map lattice is given. An immediate consequence is Theorem 1.2 and this description is also used in the proofs of other main results. In particular, Theorem 1.3 is proved in Section 4 using additional consequences of the piecewise monotonicity of \( f \) and positivity of the coupling operator. Finally, the proof of Proposition 1.1 and the existence of symmetric orbits is given in Section 5.

1.2 Dynamics of unimodal maps with simple kneading sequences

The dynamics of a unimodal map is largely determined by its kneading sequence, i.e. the location of the iterates of the point 1. Our study of coupled maps also relies on such information from the local map.

The itinerary \( I(x) = \{I_i(x)\}_{i \in \mathbb{N}} \) of a point \( x \in [-1, 1] \) is either an infinite sequence of \( L \)'s and \( R \)'s, or a finite sequence of \( L \)'s and \( R \)'s, followed by \( C \). The \( i \)-th element of \( I(x) \) is defined by

\[
I_i(x) = \begin{cases} 
L & \text{if } f^i(x) < 0 \\
C & \text{if } f^i(x) = 0 \\
R & \text{if } f^i(x) > 0 
\end{cases}
\]

The kneading sequence of a unimodal map is the itinerary of the point 1.

The simplest kneading sequences are \( I^n := R^{*n} \star R^\infty \) (\( A^\infty \) denotes the infinite concatenation \( AAA \cdots \)) which is \( 2^n \)-periodic and \( J^n := R^{*n} \star R C \) which is finite. These sequences are well-known (see for instance [4] for their definitions using the *-product). They can also be directly defined by induction (see Appendix A for the inductive definition and simple properties). When the kneading sequence is \( I^n \) or \( J^n \), the asymptotic dynamics of \( f \) is quite simple as claimed in the next statement (see Section 5.1 for the proof).

In the rest of the paper, we assume that the kneading sequence of \( f \) is either \( I^n \) or \( J^n \) for some \( n \in \mathbb{N} \). (The integer \( n \) then always refers to the kneading sequence.)

**Proposition 1.1** (1) For every \( x \in [f(1), 1] \), there exists \( k \in \{0, \ldots, n+1\} \) such that

\[
\lim_{i \to \infty} f_i(x) - f_{i+2^k}(x) = 0.
\]
(2) For every $k \in \{0, \cdots, n\}$, there exists a unique $2^k$-periodic point, $x_k$, between 0 and $f^{2^k}(0)$.

(3) If $f$ is in addition $S$-unimodal and $n \geq 1$, then for each $k \in \{0, \cdots, n-1\}$ (and possibly for $k = n$ when $n \in \mathbb{N}$), this $2^k$-periodic orbit is unstable with $(f^{2^k})'(x_k) < -1$.

For instance, for every $a$ smaller than the accumulation point of the period-doubling cascade, the kneading sequence of the quadratic map $f_a$ is either $P^n$ or $J^n$ for some $n \in \mathbb{N}$ (which depends on $a$).

1.3 Dynamics of the coupled map lattice

As we mentioned in the introduction, analytic and numerical studies of two coupled quadratic maps have revealed the presence of Hopf bifurcations and thus, the possible existence of quasi-periodic orbits. Therefore, one cannot expect every orbit of $F_c$ to be asymptotically periodic. Nevertheless, one can prove eventual periodicity of the position with respect to the diagonal, i.e. of the sign of $x^t - y^t$.

In the next theorem, we only consider orbits for which $x^t \neq y^t$ for all $t \in \mathbb{N}$. Otherwise, $x^t - y^t$ is eventually zero (and hence eventually periodic).

**Theorem 1.2** For every $(x, y) \in \mathcal{M}$ such that $x^t \neq y^t$ for all $t \in \mathbb{N}$, there exist $t_0 \in \mathbb{N}$ and $k \in \{0, \cdots, n\}$ so that for all $t \geq t_0$, we have

\[
(x^{t+2^k} - y^{t+2^k})(x^t - y^t) < 0. \tag{1.1}
\]

All cases $k \in \{0, \cdots, n\}$ can occur. Moreover, the relation (1.1) implies that every orbit satisfies the following property (which may be useful for a stroboscopic plotting of the dynamics). There exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, we have

\[
(x^{t+2^{n+1}} - y^{t+2^{n+1}})(x^t - y^t) \geq 0.
\]

Not only the position with respect to the diagonal is eventually periodic, but also the orbit itself can be asymptotically periodic. As claimed in the next statement, this happens when the coordinates $x^t$ and $y^t$ always have the same sign.

**Theorem 1.3** Let $\{(x^t, y^t)\}_{t \in \mathbb{N}}$ be an orbit such that $x^t y^t \geq 0$ for all $t \in \mathbb{N}$. There exists $T \in \{0, \cdots, n+1\}$ such that

\[
\lim_{t \to \infty} x^{t} - x^{t+2T} = \lim_{t \to \infty} y^{t} - y^{t+2T} = 0.
\]
Moreover, if the orbit is not asymptotically homogeneous, then \( T = k + 1 \) where \( k \) is the integer in Theorem 1.2 for which the relation (1.1) holds.

The condition \( x^t y^t \geq 0 \) for all \( t \) is not only a technical condition emerging from our proof but also a natural condition. Indeed, the Hopf bifurcations mentioned before, from which quasi-periodic orbits may be created, occur for orbits with different signs coordinates, i.e., for which \( x^t y^t < 0 \) for some \( t \) [5].

In the introduction, we mentioned that every orbit is asymptotically homogeneous when the coupling parameter \( \epsilon \) is sufficiently close to \( \frac{1}{4} \). (As a consequence, the mentioned Hopf bifurcations must occur when \( \epsilon \) is far from \( \frac{1}{4} \).) If \( n \geq 1 \), the CML has non-homogeneous periodic orbits when \( \epsilon \) is sufficiently small. Furthermore, a bifurcation analysis of the homogeneous \( 2^k \)-periodic orbit passing through the point \( (x_k, x_k) \) (see Proposition 1.1-(3)) shows that when its normal multiplier (i.e., the eigenvalue of the derivative \( DF^f_{x_k}(x_k, x_k) \) in the direction perpendicular to the diagonal), \( (1 - 2\epsilon)^{2^k} (f^{2^k})'(x_k) \), crosses \(-1\), we have a period doubling bifurcation creating a symmetric \( 2^{k+1} \)-periodic orbit [5]. On the other hand, if \( n \geq 1 \), symmetric orbits also exist when \( \epsilon = 0 \). By symmetry of \( F \), and uniqueness in Implicit Function Theorem, when these symmetric orbits are hyperbolic, they persist for small \( \epsilon \). The next statement tells us that such symmetric orbits exist for every coupling parameter between the period doubling bifurcation point and 0.

**Theorem 1.4** Let \( x_k \) where \( k \in \{0, \cdots, n\} \), be the \( 2^k \)-periodic point of \( f \) between 0 and \( f^{2^k}(0) \). If

\[
(1 - 2\epsilon)^{2^k} (f^{2^k})'(x_k) < -1
\]

then \( F \) has a symmetric \( 2^{k+1} \)-periodic orbit.

In particular, if \( f \) is \( S \)-unimodal, then by Proposition 1.1-(3), for each \( k \in \{0, \cdots, n - 1\} \) (and possibly \( k = n \)), condition (1.2) holds if \( \epsilon \) is suitably chosen close to 0.

## 2 Dynamical properties of unimodal maps

In the analysis of the coupled system, we need to use some properties of the iterates of some special triangles of the phase space. Given \( a, b \in [-1, 1] \), \( a < b \), let

\[
\mathcal{T}_{a, b} = \{(x, y) \in [-1, 1]^2 : a \leq x \leq b \text{ and } x \leq y \leq b\}
\]

denote a triangle above the diagonal and let \( \mathcal{T}_{a, b} = \theta(\mathcal{T}_{a, b}) \). Let \( S_{a, b} = \mathcal{T}_{a, b} \cup \mathcal{T}_{a, b}^c \), where \( I_{a, b} = [a, b] \). For convenience, let \( U_{\{a, b\}} = U_{\min\{a, b\}, \max\{a, b\}} \)

where \( U \) is any of the symbols \( \mathcal{T}, \mathcal{T}^c, S, \) or \( I \).
Given a unimodal map $f$, assume that, for some $a \in [-1, 1]$ and some $t \in \mathbb{N}$, the iterates $f^i(a)$ and $f^t(0)$, where $i \in \{1, \ldots, t\}$, lie on the same side of $0$. Then $f^{t+1}(I_{(0,a)}) = I_{(f^{t+1}(0), f^{t+1}(a))}$ and the relative positions of $f^{t+1}(a)$ and $f^{t+1}(0)$ depend on the number of times when $f^i(a)$ is positive. In the next proposition, we present an alternative formulation of this property which will be used several times in later proofs. Recall that a finite sequence of $L$'s and $R$'s is said to be odd if it has an odd number of $R$'s, even, otherwise.

**Proposition 2.1** Let $a \in [-1, 1] \setminus \{0\}$ and $t \in \mathbb{N}$ be such that $\{I_i(a)\}_{1 \leq i \leq t} = \{I_{i-1}(1)\}_{1 \leq i \leq t}$ if $t \geq 1$. If neither of these two sequences contains the symbol $C$, then we have

1. $F_0^{t+1}(\mathcal{T}_{(0,a)}) = \theta^s(\mathcal{T}_{(f^{t+1}(0), f^{t+1}(a))})$ and
2. $(-1)^s(a(f^{t+1}(a) - f^{t+1}(0)) > 0$,

where $s = 0$ (resp. $s = 1$) if the sequence $\{I_i(a)\}_{0 \leq i \leq t}$ is even (resp. odd).

**Proof:** We use induction on $t$. For $t = 0$, the results follow from the fact that $f$ is a bijection from $I_{(0,a)}$ to $I_{(f(a),1)}$ which preserves (resp. reverses) orientation if $a < 0$ (resp. $a > 0$).

Now, given $t \in \mathbb{N}$, assume that (i) and (ii) and suppose that

$$\{I_i(a)\}_{1 \leq i \leq t+1} = \{I_{i-1}(1)\}_{1 \leq i \leq t+1},$$

where all symbols are different from $C$. We first consider the case when the sequence $\{I_i(a)\}_{0 \leq i \leq t+1}$ is even. Then, either $\{I_i(a)\}_{0 \leq i \leq t}$ is even and $I_{t+1}(a) = L$, or $\{I_i(a)\}_{0 \leq i \leq t}$ is odd and $I_{t+1}(a) = R$. In the first situation, both $f^{t+1}(0) = f^t(1)$ and $f^{t+1}(a)$ are negative. Since $f$ is continuous and strictly increasing on $[-1,0]$, we have $(f^{t+1}(a) - f^{t+1}(0))(f^{t+2}(a) - f^{t+2}(0)) > 0$. Using the induction hypothesis, it then follows that

$$F_0^{t+2}(\mathcal{T}_{(0,a)}) = F_0(\mathcal{T}_{(f^{t+1}(0), f^{t+1}(a))}) = \mathcal{T}_{(f^{t+2}(0), f^{t+2}(a))},$$

and $a(f^{t+2}(a) - f^{t+2}(0)) > 0$ because $a(f^{t+1}(a) - f^{t+1}(0)) > 0$. Similarly, in the second situation when $I_{t+1}(a) = R$, since both $f^{t+1}(0)$ and $f^{t+1}(a)$ are positive, we have

$$F_0^{t+2}(\mathcal{T}_{(0,a)}) = F_0(\mathcal{T}_{(f^{t+1}(0), f^{t+1}(a))}) = \mathcal{T}_{(f^{t+2}(0), f^{t+2}(a))},$$

and again $a(f^{t+2}(a) - f^{t+2}(0)) > 0$ because now $a(f^{t+1}(a) - f^{t+1}(0)) < 0$. Thus, both statements (i) and (ii) hold for $t + 1$ in the case when the sequence $\{I_i(a)\}_{0 \leq i \leq t+1}$ is even. For the case when the sequence $\{I_i(a)\}_{0 \leq i \leq t+1}$ is odd, the proof follows analogously. □
When the kneading sequence of $f$ is either $I^n$ and $J^n$, the properties of the iterates $\{f^{2^n}(0)\}$ were extensively discussed in [4]. In the next proposition, we collect the ones needed to prove the first two main results.

**Proposition 2.2** The following statements hold for every $k \in \{0, \cdots, n\}$.

1. $f^{2^k}(0) > 0$ (resp. $f^{2^k}(0) < 0$) if $k$ is even (resp. odd).
2. If $k \geq 1$, then $\{I_i(f^{2^k}(0))\}_{1 \leq i \leq 2^n-1} = \{I_{i-1}(1)\}_{1 \leq i \leq 2^n-1}$ and neither of these two sequences contains the symbol $C$.
3. The sequence $\{I_i(f^{2^k}(0))\}_{0 \leq i \leq 2^n-1}$ is odd.
4. $S_{\{f^{2^n}(0), f^{2^{n+1}}(0)\}} \subset S_{\{0, f^{2^n}(0)\}}$.

We postpone the proof to Appendix A. Additional properties are given in Lemma 5.2 when they are needed.

### 3 Dynamical properties of the CML - Proof of Theorem 1.2

Using the results of the previous section together with the convexity of the coupling operator $L$, we obtain a precise description of the iterates of any point in $\mathcal{M}$ under the coupled map lattice, **Proposition 3.1** below. In particular, Theorem 1.2 immediately follows from this statement when one considers those $(x, y) \in \mathcal{M}$ for which $x^t \neq y^t$ for all $t \in \mathbb{N}$.

**Proposition 3.1** For every $(x, y) \in \mathcal{M}$, there exists $k \in \{0, \cdots, n\}$ and $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$ and $p \in \{0, \cdots, 2^k - 1\}$, there exists $\sigma(t, p) \in \mathbb{N}$ with $\sigma(t+1, p) = \sigma(t, p) + 1$ so that

\[
(x^{2^k t + p}, y^{2^k t + p}) \in \Theta^{\sigma(t, p)}(T_{\{f^{2^n}(0), f^{2^{n+1}}(0)\}}).
\]

If $k \in \{0, \cdots, n-1\}$ and $p \in \{0, \cdots, 2^k - 1\}$,

\[
(x^{2^k t + p}, y^{2^k t + p}) \in \Theta^{\sigma(t, p)}(T_{\{f^{p}(0), f^{2^n}(0)\}}),
\]

if $k = n$.

In the proof, we use the following properties of the dynamics of certain triangles $T_{\{a, b\}}$. 

8
Lemma 3.2  (1) For every \( t \in \{1, \cdots, 2^n\} \), there exists \( s(t, n) \in \{0, 1\} \) such that

\[
F^t_e(\mathcal{T}_{f^0, f^{2^n}(0)}) \subset \Theta^{s(t, n)}(\mathcal{T}_{f^0, f^{2^n+1}(0)}).
\]

(2) If \( n \geq 1 \), then for every \( k \in \{0, \cdots, n - 1\} \) and every \( t \in \{1, \cdots, 2^{k+1}\} \), there exists \( s(t, k) \in \{0, 1\} \) such that for any \((x, y) \in \mathcal{T}_{f^{2^k}(0), f^{2^{k+1}}(0)}\), at least one of the following relations holds.

(i) \( F^{t}_{e}(x, y) \in \Theta^{s(t, k)}(\mathcal{T}_{f^{t}(0), f^{2^{k+1}+1}(0)}) \).

(ii) \( F^{t}_{e}(x, y) \in \Theta^{s(t, k+1)}(\mathcal{T}_{f^{t}(0), f^{2^{k+1}+1}(0)}) \).

(3) \( s(2^k, k) = 1 \) for every \( k \in \{0, \cdots, n\} \) in all relations.

Proof of the Lemma: (1) We will actually prove the following stronger statement. For every \( k \in \{0, \cdots, n\} \) and every \( t \in \{1, \cdots, 2^k\} \), there exists \( s(t, k) \in \{0, 1\} \) such that

\[
F^t_e(\mathcal{T}_{f^0, f^{2^k}(0)}) \subset \Theta^{s(t, k)}(\mathcal{T}_{f^0, f^{2^{k+1}}(0)}).
\]  

(3.1)

Let \( k \in \{0, \cdots, n\} \). By Proposition 2.2-(2), one can apply Proposition 2.1-(i) with \( a = f^{2^k}(0) \) to obtain the existence, for every \( t \in \{1, \cdots, 2^k\} \), of \( s(t, k) \in \{0, 1\} \) such that

\[
F^t_e(\mathcal{T}_{f^0, f^{2^k}(0)}) = \Theta^{s(t, k)}(\mathcal{T}_{f^0, f^{2^{k+1}}(0)}).
\]  

(3.2)

(In the case where \( k = 0 \), this relation is obvious.)

On the other hand, since \( \epsilon \in \left[0, \frac{1}{2}\right] \), for any \( a, b \in [-1, 1] \) and any \( s \in \{0, 1\} \), we have

\[
L_{\epsilon}(\Theta^s(\mathcal{T}_{\{a, b\}})) \subset \Theta^s(\mathcal{T}_{\{a, b\}}).
\]  

(3.3)

We now use induction on \( t \). Using the decomposition \( F_{e} = L_{\epsilon} \circ F_{0} \) and the relations (3.2) and (3.3), we obtain \( F_{e}(\mathcal{T}_{f^0, f^{2^k}(0)}) \subset \Theta^{s(1, k)}(\mathcal{T}_{f^0, f^{2^{k+1}}(0)}) \).

That is to say, the relation (3.1) holds for \( t = 1 \).

Assume now that this relation holds for some \( t \in \{1, \cdots, 2^k - 1\} \). Then by applying the relations (3.2) and (3.3), we obtain

\[
F_{e}^{t+1}(\mathcal{T}_{f^0, f^{2^k}(0)}) \subset L_{\epsilon}(\Theta^{s(t, k)}(\mathcal{T}_{f^0, f^{2^{k+1}}(0)}))
\]

\[
= L_{\epsilon}(s(t+1, k)(\mathcal{T}_{f^{t+1}(0), f^{2^{k+1}+1}(0)}))
\]

\[
\subset \Theta^{s(t+1, k)}(\mathcal{T}_{f^{t+1}(0), f^{2^{k+1}+1}(0)}),
\]

showing that the relation (3.1) holds for \( t + 1 \) and hence holds for all \( t \in \{1, \cdots, 2^k\} \).
(2) We note that by definition of \( f \), for any \( a, b \in [-1, 1] \) such that \( a < 0 < b \), we have
\[
F_0(\overline{T}_{a,b}) \subset \{(x, y) \in \mathcal{M} : f(a) \leq x \leq 1 \text{ and } f(b) \leq y \leq 1\}
\]
\[
\subset F_0(\overline{T}_{a,b}) \cup F_0(\overline{T}_{0,b}) = \overline{T}_{f(a),1} \cup \mathcal{U}_{f(b),1}.
\]
Using the relations (3.2) and (3.3), we obtain the following inclusion for any \( k \in \{0, \ldots, n - 1\} \)
\[
F_e(\overline{T}_{f^{2^k}(0), f^{2^{k+1}}(0)}) \subset \theta^{(1,k)}(\overline{T}_{f^{2^{k+1}}(0),1}) \cup \theta^{(1,k+1)}(\overline{T}_{f^{2^{k+1}}+1(0),1}).
\]
Consider the case when
\[
F_e(x, y) \in \theta^{(1,k)}(\overline{T}_{f^{2^{k+1}}(0),1}) = F_0(\overline{T}_{0,f^{2^k}(0)}).
\]
Using the same argument as that in proving the relation (3.1), we obtain by (3.2) that for every \( t \in \{1, \ldots, 2^k\} \)
\[
F_e^{t-1}(\theta^{(1,k)}(\overline{T}_{f^{2^{k+1}}(0),1})) \subset F_0^{t-1}(\theta^{(1,k)}(\overline{T}_{f^{2^{k+1}}(0),1})) = \theta^{(t,k)}(\overline{T}_{f^{t}(0), f^{2^k+1}(0)}).
\]
Therefore, the relation (i) follows. In the case when
\[
F_e(x, y) \in \theta^{(1,k+1)}(\overline{T}_{f^{2^{k+1}}+1(0),1}),
\]
because \( k < n \), the proof is exactly the same by replacing \( k \) with \( k + 1 \) to obtain the relation (ii).

(3) Finally, the fact that \( s(2^k, k) = 1 \) is a consequence of Propositions 2.2-(3) and 2.1.

□

**Proof of Proposition 3.1.** We consider two cases, namely \((x, y) \in S_{(0, f^{2^n}(0))}\) and \((x, y) \not\in S_{(0, f^{2^n}(0))}\).

(a) Assume that \((x, y) \in S_{(0, f^{2^n}(0))}\), i.e., \((x, y) \in \theta^n(\overline{T}_{(0, f^{2^n}(0))})\) for some \( s \in \{0, 1\} \). Applying Lemma 3.2-(1) with \( t = 2^n \) and using Proposition 2.2-(4), it follows that
\[
(x^{2^n}, y^{2^n}) \in \theta^{s(2^n, n)}(\overline{T}_{(0, f^{2^n}(0))}).
\]
Repeating this argument in an induction, we obtain
\[
(x^{2^m}, y^{2^m}) \in \theta^{s(m)(2^n, n)}(\overline{T}_{(0, f^{2^n}(0))}),
\]
for all \( m \in \mathbb{N} \). For any given \( p \in \{1, \ldots, 2^n - 1\} \), applying again Lemma 3.2-(1) with \( t = p \) to the point \((x^{2^m}, y^{2^m})\), we have
\[
(x^{2^m+p}, y^{2^m+p}) \in \theta^{s(2^m+p)+(p, n)}(\overline{T}_{(f^p(0), f^{2^n+p}(0))}).
\]

10
for all \( m \in \mathbb{N} \). The result follows with \( k = n \), \( t_0 = 0 \) and \( \sigma(t, p) = s + ts(2n, n) + s(p, n) \) (when letting \( s(p, n) = 0 \) if \( p = 0 \)).

(b) Assume that \((x, y) \notin S_{\{0, f^n(0)\}}\). Since \( M = S_{f^n(0), f^n(0)} \), there exists \( k' \in \{0, \cdots, n - 1\} \) such that \((x, y) \in S_{\{f^{k'}(0), f^{k'+1}(0)\}}\). Let \( j \) be the largest of such \( k' \). Then \((x, y) \in \theta^s(T_{\{f^{j}(0), f^{j+1}(0)\}}) \) for some \( s \in \{0, 1\} \).

Applying Lemma 3.2-(2) with \( k = j \) and \( t = 2^j \) to \((x, y)\) and, repeatedly, to the subsequent iterates, we have either
\[
(x^{2^m}, y^{2^m}) \in \theta^{s+m}s(2^j, j) (T_{\{f^{2^j}(0), f^{2^j+1}(0)\}}),
\]
for all \( m \in \mathbb{N} \), or there exists \( m' \in \mathbb{N} \) such that
\[
(x^{2^m'}, y^{2^m'}) \notin \theta^{s+m's(2^j, j)} (T_{\{f^{2^j}(0), f^{2^j+1}(0)\}}).
\]

The latter situation occurs when \((x^{2^m' - 1}, y^{2^m' - 1})\) does not satisfy the property (i) of Lemma 3.2-(2).

In the first case, using the same argument as that in the case (a), we conclude that
\[
(x^{2^m+p}, y^{2^m+p}) \in \theta^{s+m's(2^j, j)} (T_{\{f^{2^j}(0), f^{2^j+p}(0)\}}),
\]
for all \( p \in \{1, \cdots, 2^j - 1\} \). Thus, the result holds with \( k = j \), \( t_0 = 0 \) and \( \sigma(t, p) = s + ts(2^j, j) + s(p, j) \).

In the second case, the property (ii) of Lemma 3.2-(2) with \( k = j \) and \( t = 2^{j+1} \) must apply to the point \((x^{2^m'-1}, y^{2^m'-1})\). It results that
\[
(x^{2^m' - 1}, y^{2^m' - 1}) \in S_{\{f^{2^j+1}(0), f^{2^j+2}(0)\}}.
\]

If \( j = n - 1 \), then by Proposition 2.2-(4), we have \((x^{2^m' - 1}, y^{2^m' - 1}) \in S_{\{0, f^n(0)\}}\) and we are back in the case (a) with \((x^{2^m' - 1}, y^{2^m' - 1})\) instead of \((x, y)\). We conclude that the result holds with \( k = n \) and \( t_0 = m' + 1 \).

Now, if \( j \leq n - 2 \), we apply Lemma 3.2-(2) with \( k = j + 1 \) and \( t = 2^{j+1} \) to the point \((x^{2^m' - 1}, y^{2^m' - 1})\) and, repeatedly, to the subsequent iterates. We conclude that either the result holds with \( k = j + 1 \) and \( t_0 = m' + 1 \) or there exists \( m'' \in \mathbb{N} \) such that \((x^{2^m' + 2^{j+1}m''}, y^{2^m' + 2^{j+1}m''}) \) does not belong to \( \theta^{s+(m' + 1)s(2^j, j) + m''}(T_{\{f^{2^j+2}(0), f^{2^j+3}(0)\}}) \). Again, we see that if \( j = n - 2 \), then the point
\[
(x^{2^m' + 2^{j+1}(m'' + 1)}, y^{2^m' + 2^{j+1}(m'' + 1)})
\]
belongs to \( S_{\{0, f^n(0)\}} \) and the result holds. If \( j \leq n - 3 \), we need to apply repeatedly Lemma 3.2-(2) with \( k = j + 2 \) and \( t = 2^{j+2} \) to this point and so on. This process is finite and it ends within at most \( n \) steps. We conclude that the result holds for some \( k \in \{j, \cdots, n\} \) and \( t_0 \in \mathbb{N} \). □
4 Proof of Theorem 1.3

Let \( \{(x^t, y^t)\}_{t \in \mathbb{N}} \) be an orbit such that \( x^t y^t \geq 0 \) for all \( t \in \mathbb{N} \). If \( x^{t'} = y^{t'} \) for some \( t' \in \mathbb{N} \), then \( x^{t+1} = y^{t+1} = f(x^t) \) for all \( t \geq t' \) and the result is given by Proposition 1.1.-1. Therefore, in all the proof, we also assume that \( x^t \neq y^t \) for all \( t \).

Let \( k \in \{0, \cdots, n\} \) and let \( t_0 \in \mathbb{N} \) be integers for which Proposition 3.1 holds. We claim that for all \( t \geq t_0 \), we have

\[
(x^{2^{k+1} t}, y^{2^{k+1} t}) \in S_{\{0, f^{2^{k+1}(0)}\}}, \tag{4.1}
\]

This is immediate if \( k = n \). If \( k < n \), then by assumption on the orbit, we have

\[
(x^{2^{k+1} t}, y^{2^{k+1} t}) \in S_{\{0, f^{2^{k+1}(0)}\}} \cup S_{\{0, f^{2^{k+1+1}(0)}\}}.
\]

(By Proposition 2.2.-1, these squares only intersect at \( (0, 0) \).) By contradiction, if otherwise \( (x^{2^{k+1} t'}, y^{2^{k+1} t'}) \in S_{\{0, f^{2^{k+1+1}(0)}\}} \setminus S_{\{0, f^{2^{k+1}(0)}\}} \) for some \( t' \in \mathbb{N} \), then by the relation (3.1) with \( t = 2^{k+1} \), we would have

\[
(x^{2^{k}(t'+2)} - y^{2^{k}(t'+2)}) (x^{2^{k} t'} - y^{2^{k} t'}) < 0
\]

But we assumed that Proposition 3.1 holds with \( k \). So

\[
(x^{2^{k}(t'+2)} - y^{2^{k}(t'+2)}) (x^{2^{k} t'} - y^{2^{k} t'}) > 0
\]

and we have a contradiction.

We now show that every orbit satisfying the relation (4.1) is asymptotically periodic with period a factor of \( 2^{k+1} \). Without loss of generality, we may assume that \( t_0 = 0 \). We may also assume that \( (x^{2^{k+1}}, y^{2^{k+1}}) \neq (x, y) \). Otherwise, there is nothing to prove. For each point \((a, b) \in \mathcal{M}\), we introduce the notation \( (a, b) \prec 0 \) which means that both \( \max\{a, b\} \leq 0 \) and \( \min\{a, b\} < 0 \) and we divide all possible situations of \((x^{2^{k+1}}, y^{2^{k+1}}) \neq (x, y)\) into two cases:

(a) \((-1)^r (x^{2^{k+1}} - x, y^{2^{k+1}} - y) \prec 0 \) for some \( r \in \{0, 1\} \).
(b) \((-1)^r (x^{2^{k+1}} - x, y - y^{2^{k+1}}) \prec 0 \) for some \( r \in \{0, 1\} \).

In the first case, we use the following Lemma whose proof is given in Appendix B.

**Lemma 4.1** Let \( a \in [-1, 1] \setminus \{0\} \) and \( t \in \mathbb{N} \) be such that \( \{I_i(a)\}_{1 \leq i \leq t} = \{I_{i-1}(1)\}_{1 \leq i \leq t} \) if \( t \geq 1 \). If neither of these two sequences contains the symbol \( C \), then for every pair of points \((x_1, y_1), (x_2, y_2) \in S_{\{0, a\}} \) with \( (x_1 - x_2, y_1 - y_2) \prec 0 \), we have

\[
(-1)^s (x_1^{t+1} - x_2^{t+1}, y_1^{t+1} - y_2^{t+1}) \prec 0
\]

where \( s = 0 \) (resp. \( s = 1 \)) if \( \{I_i(a)\}_{0 \leq i \leq t} \) is even (resp. odd).
By the relation (4.1) and by Propositions 2.2-(2) and 2.2-(3), we apply Lemma 4.1 first to \((x^{2^k}, y^{2^k}), (x, y)\) with \(a = f^{2^k}(0)\) and \(t = 2^k - 1\) to obtain
\[
(-1)^{r+1} \left( x^{2^k + 2^k} - x^{2^k}, y^{2^k + 1} - y^{2^k} \right) < 0.
\]
Applying the lemma to the points \((x^{2^k + 2^k}, y^{2^k + 1}), (x^2, y^2)\) again with \(a = f^{2^k}(0)\) and \(t = 2^k - 1\), we obtain
\[
(-1)^r \left( x^{2^k + 2^k + 1} - x^{2^k + 2^k}, y^{2^k + 2^k + 1} - y^{2^k + 2^k} \right) < 0.
\]
Repeating this process, we have for all \(t \in \mathbb{N}\)
\[
(-1)^r \left( x^{2^k + 1} - x^{2^k + 1}, y^{2^k + 1} - y^{2^k + 1} \right) < 0.
\]
Thus, both sequences \(\{x^{2^k}\}\) and \(\{y^{2^k}\}\) are monotonic. Since they are bounded, they consequently converge. By continuity of \(f\), the limit is a periodic point of \(F_c\) whose period must be a factor of \(2^{k+1}\).

In the case (b), the proof uses another lemma whose proof is also given in Appendix B. Note that if \(\varepsilon = \frac{1}{2}\), then \(x^1 = y^1\) for every \((x, y) \in M\) and thus, the case (b) cannot occur.

**Lemma 4.2** Let \(a \in [-1, 1] \setminus \{0\}\) and \(t \in \mathbb{N}\) be such that \(\{I_i(a)\}_{1 \leq i \leq t} = \{I_{i-1}(1)\}_{1 \leq i \leq t}\) if \(t \geq 1\). If neither of these two sequences contains the symbol \(C\), then for every \(\varepsilon \in [0, \frac{1}{2})\) and every pair of points \((x_1, y_1), (x_2, y_2) \in S_{[0, a)}\) with \((x_1 - x_2, y_2 - y_1) < 0\), at least one of the following statements hold.

(i) there exists \(s \in \{0, 1\}\) (which depends on \((x, y)\) and on \(a\)) such that
\[
(-1)^s(x_1^{t+1} - x_2^{t+1}, y_1^{t+1} - y_2^{t+1}) < 0.
\]

(ii) there exists \(s \in \{0, 1\}\) depending only on \(a\) such that
\[
(-1)^s(x_1^{t+1} - x_2^{t+1}, y_2^{t+1} - y_1^{t+1}) < 0.
\]

In the case (b), using again the relation (4.1) and Propositions 2.2-(2) and 2.2-(3), we apply Lemma 4.2 with \(a = f^{2^k}(0)\) and \(t = 2^k - 1\). If the statement (i) holds, then we are in the same situation as in the case (a) with \((x^{2^k}, y^{2^k})\) instead of \((x, y)\) and the orbit is asymptotically periodic with period a factor \(2^{k+1}\).

If the statement (i) does not hold, then we apply once again Lemma 4.2 with \(a = f^{2^k}(0)\) and \(t = 2^k - 1\) to conclude that either we are in the case (a) with \((x^{2^k + 1}, y^{2^k + 1})\) instead of \((x, y)\) or
\[
(-1)^r \left( x^{2^k + 1} - x^{2^k + 1}, y^{2^k + 1} - y^{2^k + 1} \right) < 0.
\]
since $s$ depends only on $k$ and $(-1)^{2s} = 1$.

By induction on $t$, we are either in the case (a) after a finite number of iterations and therefore, the orbit is asymptotically periodic with period a factor $2^{k+1}$ or we have for all $t \in \mathbb{N}$

$$(-1)^t (x^{2^{k+1}(t+1)} - x^{2^k t} - y^{2^{k+1}(t+1)} - y^{2^k t}) < 0.$$ 

In the latter case, it also follows that both sequences $\{x^{2^{k+1}t}\}$ and $\{y^{2^{k+1}t}\}$ converge and the orbit is asymptotically periodic with period a factor $2^{k+1}$ which is the desired conclusion.

To finish the proof of Theorem 1.3, it remains to show that the period of every non-homogeneous periodic orbit passing $S_{\{0,fx^k(0)\}}$ is also a multiple of $2^{k+1}$.

To see this, recall that the relation (3.1) imposes that every orbit such that $x^t \neq y^t$ and $(x^{2^k t}, y^{2^k t}) \in S_{\{0,fx^k(0)\}}$ for all $t \in \mathbb{N}$ has the property

$$(x^{2^k(t+1)} - y^{2^k(t+1)})(x^{2^k t} - y^{2^k t}) < 0. \quad (4.2)$$

By contradiction, assume now that there exists a non-homogeneous periodic orbit passing $S_{\{0,fx^k(0)\}}$ with period $2^j$ where $i \in \{0, \cdots, k\}$ and $j$ is odd. By the relation (4.2) and because $j$ is odd, we obtain that $(x^{2^j} - y^{2^j})(x - y) < 0$. But, on the other hand, $2^j$ is a multiple of $2^j$ and then we have by periodicity $(x^{2^j} - y^{2^j})(x - y) = (x - y)^2 > 0$ which leads to a contradiction. \hfill \Box

5 Existence of symmetric orbits

In this section, we will need the following additional property of the sequence $\{f^{2^k}(0)\}$ when the kneading sequence of $f$ is either $I^n$ and $J^n$.

**Proposition 5.1** For every $k \in \{1, \cdots, n\}$, we have $I_{\{f^{2^k-1}(0),f^{2^k-1+2^k}(0)\}} \subsetneq I_{\{0,f^{2^k-1}(0)\}}$.

To prove this inclusion, we use the next statement.

**Lemma 5.2** Assume that $n \geq 1$ and let $k \in \{1, \cdots, n\}$.

(1) We have $f^{2^k-1+2^k}(0) > 0$ (resp. $f^{2^k-1+2^k}(0) < 0$) if $k$ is odd (resp. $k$ is even).

(2) The sequence $\{I(I(f^{2^k}(0)))\}_{0 \leq t \leq 2^{k-1}-1}$ is even.
Proof of the Lemma: (1) We have $I_0(f^{2^{k-1}+2^k}(0)) = I_{2^{k-1}+2^k-1}(1)$ and $2^{k-1} + 2^k - 1 \leq 2^{n+1} - 2$ for all $n \geq k \geq 1$. Therefore, by Lemmas A.1-(B) and (C) in Appendix A, we need only to show that $P_n^{2^{k-1}+2^k-1} = R$ when $n$ is odd and $P_n^{2^{k-1}+2^k-1} = L$ when $n$ is even. Since the sequence $P_n$ is $2^n$-periodic, we need to prove that $P_n^{2^{k-1}+2^k-1} = R$ when $n$ is odd and $P_n^{2^{k-1}+2^k-1} = L$ when $n$ is even. As in the proof of Lemma A.1-(A), these can be easily done by induction on $n$ because we have $I_0^1 = R$ and $I_{2^{k-1}+2^k-1}^n = I_{2^{k-1}+2^k-1}^n$.

(2) Assume that the kneading sequence of $f$ is $I^k$, the other cases follow from the fact that the proof only uses the first $2^{k-1}$ symbols of the kneading sequence and Lemmas A.1-(B) and (C).

By a change of the index, we have $\{I_i(f^{2^k}(0))\}_{1 \leq i \leq 2^{k-1}-1} = \{I_i^k\}_{0 \leq i \leq 2^{k-1}-2}$. Because $f^{2^k}(0) > 0$ when $k$ is even and $f^{2^k}(0) < 0$ otherwise, to prove that $\{I_i(f^{2^k}(0))\}_{0 \leq i \leq 2^{k-1}-1}$ is even for every $k$, we only need to prove that the sequence $\{I_i^k\}_{0 \leq i \leq 2^{k-1}-2}$ is odd when $k$ is even and even when $k$ is odd. We use induction on $k$. The sequence $\{I_i^k\}_{0 \leq i \leq 0} = R$ is odd and the statement holds for $k = 2$. Now we assume that the statement holds for some $k \geq 2$, $k$ even. We have

$$\{I_i^{k+1}\}_{0 \leq i \leq 2^{k+1}-2} = \{I_i^{k+1}\}_{0 \leq i \leq 2^{k+1}-1} \cup \{I_i^{k+1}\}_{0 \leq i \leq 2^{k+1}-2}.$$

By the first equality in (A.1) of Appendix A, the sequence $\{I_i^{k+1}\}_{0 \leq i \leq 2^{k+1}-1}$ is even. Moreover, we have $\{I_i^{k+1}\}_{0 \leq i \leq 2^{k+1}-2} = \{I_i^k\}_{0 \leq i \leq 2^{k-1}-2}$ which is also an even sequence by the induction assumption. Therefore, the statement holds for $k+1$ which is an odd number. A similar argument can be done in the case where $k$ is odd. This completes the proof.

Proof of Proposition 5.1. By Lemma 5.2-(1) and Proposition 2.2-(1), for every $k \in \{1, \ldots, n\}$, the points $f^{2^{k-1}}(0)$ and $f^{2^{k-1}+2^k}(0)$ lie on the same side of 0. Therefore, we only need to prove the inequality

$$f^{2^{k-1}}(0)(f^{2^{k-1}+2^k}(0) - f^{2^{k-1}}(0)) < 0. \tag{5.1}$$

By the same argument as in the proof of Lemma 5.2-(2), we only have to prove this inequality when the kneading sequence is $I^k$. This sequence is $2^k$-periodic. So we have

$$\{I_i(f^{2^k}(0))\}_{1 \leq i \leq 2^{k-1}-1} = \{I_i^{k-1}\}_{1 \leq i \leq 2^{k-1}-1} = \{I_{i-1}(1)\}_{1 \leq i \leq 2^{k-1}-1}.$$

We now apply Lemma 2.1-(ii) with $a = f^{2^k}(0)$ and $t = 2^{k-1} - 1$ and we use Lemma 5.2-(2) to obtain, the inequality $f^{2^k}(0)(f^{2^{k-1}+2^k}(0) - f^{2^{k-1}}(0)) > 0$. Thus, the fact $f^{2^{k-1}}(0)f^{2^k}(0) < 0$ implies (5.1).
5.1 Proof of Proposition 1.1

(1) Similarly as in Section 3 and using the fact that the homogeneous orbits of \(F_n\) can be identified with the orbits of \(f\), we first prove that for every point \(x \in [f(1),1]\), there exists \(k \in \{0,\cdots,n\}\) and \(t_0 \in \mathbb{N}\) such that for all \(t \geq t_0\), we have

\[ f^{2^k t}(x) \in I_{\{0,f^{2^n}(0)\}}. \tag{5.2} \]

We consider two cases, namely \(x \in I_{\{0,f^{2^n}(0)\}}\) and \(x \not\in I_{\{0,f^{2^n}(0)\}}\).

(a) If \(x \in I_{\{0,f^{2^n}(0)\}}\), then the relation (5.2) follows from case (a) in the proof of Proposition 3.1.

(b) If \(x \not\in I_{\{0,f^{2^n}(0)\}}\), then let \(j < n\) be the largest of such \(k\) so that \(x \in I_{\{0,f^{2^j}(0)\}}\). By Lemma 3.2-(2) with \(k = j\) and \(t = 2^j\), we obtain that either \(f^{2^j}(x) \in I_{\{f^{2^j}(0),f^{2^j+1}(0)\}}\) or \(f^{2^j}(x) \not\in I_{\{f^{2^j}(0),f^{2^j+1}(0)\}}\) (or both).

By Proposition 5.1, this implies that either \(f^{2^j}(x) \in I_{\{0,f^{2^j}(0)\}}\) or \(f^{2^j}(x) \not\in I_{\{0,f^{2^j}(0)\}}\) and then \(f^{2^j}(x) \in I_{\{0,f^{2^j+1}(0)\}}\).

Repetedly, we obtain that either the relation (5.2) holds for all \(t \in \mathbb{N}\) or there exists \(t' \in \mathbb{N}\) such that \(f^{2^j}(x) \in I_{\{0,f^{2^{j+1}}(0)\}}\). In the latter situation, as in the proof of Proposition 3.1, we repeat the argument with \(f^{2^j t'}(x)\) instead of \(x\) and \(j + 1\) instead of \(j\). Again this process is finite and it ends within at most \(n\) steps. Therefore, the relation (5.2) holds.

The proof is now the same as in case (a) of the proof of Theorem 1.3. We conclude that the sequence \(\{f^{2^k t}(x)\}\) is monotone and hence converges towards a periodic orbit of \(f\) whose period is a factor of \(2^k\).

(2) Let us first prove the uniqueness of \(2^k\)-periodic points in \(I_{\{0,f^{2^n}(0)\}}\) by contradiction. Assume that there exist two such points, say \(x\) and \(x^*\). It would mean that \((-1)^s(x-x^*)(x-x^*) < 0\) for some \(r \in \{0,1\}\). Then, applying Lemma 4.1 with \(a = f^{2^k}(0)\) and \(t = 2^k - 1\), we obtain \((-1)^{r+1}(x-x^*)(x-x^*) < 0\) by Proposition 2.2-(3). But this contradicts the definition of \(<\).

To prove the existence, we observe that Propositions 2.2-(1) and (4) imply that, for every \(k \in \{0,\cdots,n\}\), the product \((f^{2^{k+1}}(0) - f^{2^k}(0))(f^{2^k}(0) - 0) < 0\). By continuity, it follows that \(f^{2^k}\) has a fixed point in \(I_{\{0,f^{2^n}(0)\}}\). The proof is complete when \(k = 0\).

If \(k \geq 1\), then Propositions 2.2-(1), 2.2-(2) and 5.1 show that

\[ f^{2^{k-1}}(I_{\{0,f^{2^n}(0)\}}) \cap I_{\{0,f^{2^n}(0)\}} = I_{\{f^{2^{k-1}}(0),f^{2^{k-1}+2^n}(0)\}} \cap I_{\{0,f^{2^n}(0)\}} = \emptyset. \tag{5.3} \]

Therefore, every fixed point of \(f^{2^k}\) in \(I_{\{0,f^{2^n}(0)\}}\) is actually a \(2^k\)-periodic point of \(f\).
(3) If $f$ is $S$-unimodal, then it has at most one stable periodic orbit in $[f(1), 1]$ and this orbit attracts 0 [4]. Let us prove that, when the kneading sequence is $I^n$ or $J^n$, this orbit has period $2^n$ or $2^{n+1}$.

We first use induction on $t$ to show that $f^{2^n t}(0) \in I_{f^{2^n t}(0), f^{2^n t}(0)}$ and

$$f^{2^n (t+2)}(0) \in I_{f^{2^n (t+1)}(0), f^{2^n (t)}(0)},$$

(5.4)

for all $t \in \mathbb{N}$. For $t = 0$, the result is given by Proposition 2.2-(4). Assume now that the result holds for some $t \in \mathbb{N}$. Then, since both $f^{2^n t}(0)$ and $f^{2^n (t+2)}(0)$ belong to $I_{0, f^{2^n t}(0)}$, we can apply Lemma 4.1 to $(f^{2^n (t+2)}(0), f^{2^n (t+2)}(0))$ and $(f^{2^n t}(0), f^{2^n t}(0))$ with $a = f^{2^n t}(0)$ and $t = 2^n - 1$ to obtain using Proposition 2.2-(2)

$$(f^{2^n (t+3)}(0) - f^{2^n (t+1)}(0))(f^{2^n (t+2)}(0) - f^{2^n t}(0)) < 0.$$

Similarly, Lemma 4.1 applied to the points

$$(f^{2^n (t+2)}(0), f^{2^n (t+2)}(0)) \text{ and } (f^{2^n (t+1)}(0), f^{2^n (t+1)}(0)),$$

gives

$$(f^{2^n (t+3)}(0) - f^{2^n (t+2)}(0))(f^{2^n (t+2)}(0) - f^{2^n (t+1)}(0)) < 0.$$

These two inequalities, together with the assumption

$$f^{2^n (t+2)}(0) \in I_{f^{2^n (t+1)}(0), f^{2^n t}(0)}$$

imply

$$f^{2^n (t+3)}(0) \in I_{f^{2^n (t+2)}(0), f^{2^n (t+1)}(0)}.$$

In particular, we have $f^{2^n (t+3)}(0) \in I_{0, f^{2^n t}(0)}$ and the induction follows.

Clearly, the property (5.4) implies that the sequence $\{f^{2^n t}(0)\}_{t \in \mathbb{N}}$ is monotonic and then converges to a limit in $I_{0, f^{2^n t}(0)}$. The limit is a periodic point with period a factor of $2^{n+1}$. But $f^{2^{n+1}}(I_{0, f^{2^n t}(0)}) \cap I_{0, f^{2^n t}(0)} = \emptyset$ by (5.3). We conclude that the period of the stable periodic orbit is either $2^n$ or $2^{n+1}$ when the kneading sequence of a $S$-unimodal map is $I^n$ or $J^n$.

Moreover, for every $k \in \{0, \ldots, n\}$, the itinerary $\{I_{i(x_k)}\}_{0 \leq i \leq 2^n - 1}$ is an odd sequence by Proposition 2.2-(3). This implies that the multiplier $(f^{2^n})'(x_k)$ is negative. Since, for each $k \in \{0, \ldots, n-1\}$, the $2^k$-periodic orbit is unstable, we conclude that $(f^{2^n})'(x_k) < -1$ and the statement follows. (The same inequality may hold with $k = n$ if the stable periodic orbit has period $2^{n+1}$ which is the case when the kneading sequence is $J^n$.)
5.2 Proof of Theorem 1.4

Given \((x, y) \in \mathcal{M}\), let \(P_1(x, y) = x\) and \(P_2(x, y) = y\) denote the projections on each coordinate. Given \(k \in \mathbb{N}\), we define the functions

\[
h_k(x, y) = P_1 F^k(x, y) - y \quad \text{and} \quad g_k(x, y) = P_2 F^k(x, y) - x.
\]

The point \((x, y) \in \mathcal{M}\) with \(x \neq y\) belongs to a symmetric \(2^{k+1}\)-periodic orbit of \(F_k\) if and only if

\[
h_k(x, y) = g_k(x, y) = 0. \tag{5.5}
\]

By Proposition 3.1, every symmetric \(2^{k+1}\)-periodic orbit must pass the square \(S_{f^{2^k}(0), f^{2^{k+1}}(0)}\) \((S_{f^{2^k}(0)}\) if \(k = n\)). Indeed, let \(\{(x^i, y^i)\}_{i \in I}\) be a symmetric \(2^{k+1}\)-periodic orbit and let \(j \in \{0, \ldots, n\}\) and \(t_0 \in \mathbb{N}\) be integers for which Proposition 3.1 holds. By contradiction, we prove that \(j = k\).

If \(k < j \leq n\), then by Proposition 3.1, the points

\[
(x^{2^j t_0}, y^{2^j t_0}) \quad \text{and} \quad (x^{2^j (t_0+1)}, y^{2^j (t_0+1)})
\]

lie on different sides of the diagonal. But the assumption \(j > k\) implies that \(2^j\) is a multiple of \(2^{k+1}\) and by periodicity, these points are the same. We have a contradiction.

If \(0 \leq j < k\), then \(2^k\) is a multiple of \(2^{j+1}\). Consequently, Proposition 3.1 implies that the points \((x^{2^j t_0}, y^{2^j t_0})\) and \((x^{2^j t_0+2^k}, y^{2^j t_0+2^k})\) lie on the same side of the diagonal. But the orbit is supposed to be symmetric and \(2^{k+1}\)-periodic. Therefore, \((x^{2^j t_0+2^k}, y^{2^j t_0+2^k}) = \theta(x^{2^j t_0}, y^{2^j t_0})\) and we also have a contradiction.

Hence, we look for a solution of (5.5) in \(S_{f^{2^k}(0), f^{2^{k+1}}(0)}\) \((S_{f^{2^k}(0)}\) if \(k = n\)). The idea of the proof of the theorem is that the equations in (5.5) define \(C^1\)-curves with special relations that guarantee the existence of a common solution to both equations. Since \(g_k(x, y) = h_k(y, x)\), we concentrate on the analysis of the function \(h_k\).

Recall from Proposition 1.1-(3) that \(x_k\) denotes the unique \(2^k\)-periodic point of \(f\) between 0 and \(f^{2^k}(0)\).

**Proposition 5.3** For every \(k \in \{0, \ldots, n\}\), there exists a connected \(C^1\) curve \(H_k\) in \(S_{f^{2^k}(0), f^{2^{k+1}}(0)}\) \((S_{f^{2^k}(0)}\) when \(k = n\)), going through \((x_k, x_k)\) and joining the line segments \(x = f^{2^k}(0)\) and \(x = f^{2^{k+1}}(0)\) \((x = 0\) and \(x = f^{2n}(0)\) when \(k = n\)), on which \(h_k(x, y) = 0\).

The proof consists of three steps. We start by proving the existence of solutions of \(h_k(x, y) = 0\) in the desired squares. Then we show that these solutions form
smooth curves and finally that connected components of these curves join the
desired line segments.

Proof: For the sake of clarity, we only provide a proof in the case \(k < n\). The
proof in the case \(k = n\) is the same by replacing \(f^{2^k}(0)\) by \(f^{2^n}(0)\) and \(f^{2^{k+1}}(0)\)
by 0.

Existence of solutions: Lemma 3.2-(2) with \(t = 2^k\) implies that for every
\((x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\), the iterate \(F^{2^k}_c(x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\). (In the case
where \(k = n\), one uses Lemma 3.2-(1) to get \(F^{2^n}_c(x, y) \in S_{\{f^{2^n}(0)\}}\) for every
\((x, y) \in S_{\{f^{2^n}(0)\}}\)) Indeed, this is immediate in case (i) and it follows from
Proposition 5.1 in case (ii). In other words for every \((x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\),
we have \(P_c F^{2^k}_c(x, y) \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\). By continuity, we conclude that for
every \(x \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\), there exists \(y \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\) such that \(h_k(x, y) = 0\).

Existence of the \(C^1\)-curves: We prove that the partial derivatives of \(h_k\) are
not simultaneously zero when \(h_k(x, y) = 0\) in \(S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\).

When \(k = 0\), we have \(\frac{\partial h_0}{\partial x}(x, y) = (1 - \epsilon) f'(x).\) When \(k \geq 1\), by the chain rule, we have

\[
\frac{\partial h_k}{\partial x}(x, y) = ((1 - \epsilon) f'(x) \cdot \epsilon f'(y) \cdot z^{2^k-1}) \prod_{i=1}^{2^k-2} D F_i(x, y') \left( \frac{(1 - \epsilon) f'(x)}{\epsilon f'(x)} \right),
\]

where \(D F_i(x, y)\) denotes the Jacobian matrix computed at \((x, y)\). By Propositions 2.2-(2) and 3.2-(2), the coordinates \(x^i\) and \(y^i, i \in \{1, \ldots, 2^k - 1\}\), are either both positive or both negative for every \((x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\). Therefore, all products

\[
f'(x) \prod_{i=1}^{2^k-1} f'(x^i),
\]

where \(z^i\) can be either \(x^i\) or \(y^i\), have the same sign. The piecewise monotonicity
of \(f\) and the fact that the sequence \(\{I_i(f^{2^n}(0))\}_{0 \leq i \leq 2^n - 1}\) is odd (Proposition
2.2-(3)) imply that all these products are non-positive. Since the expression
of \(\frac{\partial h_k}{\partial x}(x, y)\) is a linear combination of terms in the form of (5.7) with non-negative coefficients, we conclude that \(\frac{\partial h_k}{\partial x}(x, y) \leq 0\) on \(S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\) and
\(\frac{\partial h_k}{\partial x}(x, y) = 0\) only if \(f'(x) = 0\) i.e. if \(x = 0\) by properties of \(f\).

Now, we consider \(\frac{\partial h_k}{\partial y}(x, y)\) when \(x = 0\) and \(y \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}\). We have
\[ \frac{\partial h_k}{\partial y}(x, y) = \epsilon f'(y) - 1 < 0 \quad \text{and when } k \geq 1 \]

\[ \frac{\partial h_k}{\partial y}(x, y) = ((1 - \epsilon)f'(x^{2^k-1}), \epsilon f'(y^{2^k-1})) \prod_{i=1}^{2^k-2} DF_i(x^i, y^i) \left( \frac{\epsilon f'(y)}{(1 - \epsilon)f'(y)} \right) - 1, \]

(5.8)

If \( x = 0 \) and \( y \in I_{f^0, f^{k+1}} \), then the same arguments as those used in the proof that \( \frac{\partial h_k}{\partial y}(x, y) \leq 0 \) show that \( \frac{\partial h_k}{\partial y}(0, y) \leq -1 \) and the two partial derivatives of \( h_k \) are not simultaneously zero.

Assume now that \( x = 0 \) and \( y \in I_{f^0, f^{k+1}} \). (Note that this case does not occur when \( k = n \).) Since \( (x, y) \in S_{f^0, f^{k+1}} \), the relation (3.1) implies that \( F^*_e(x, y) \in S_{f^0, f^{k+2k+1}} \). But Propositions 5.1 and 2.2-(1) imply that \( S_{f^k, f^{k+2k+1}} \cap S_{f^0, f^{k+1}} = \emptyset \). Consequently, there is no solution for \( h_k(x, y) = 0 \) in \( S_{f^0, f^{k+1}} \). Thus, we conclude that the partial derivatives of \( h_k \) are not simultaneously zero when \( h_k(x, y) = 0 \).

By the previous existence of solutions and Implicit Function Theorem, we conclude that for any point \( (x_0, y_0) \) not on the boundary of \( S_{f^k, f^{k+1}} \) and satisfying \( h_k(x_0, y_0) = 0 \), there is an open neighborhood of \( (x_0, y_0) \) such that its intersection with the set \( \{(x, y) : h_k(x, y) = 0\} \) is diffeomorphic to an open interval. If \( (x_0, y_0) \) is on the boundary of this square, the intersection is diffeomorphic to a semi-closed interval. This proves the existence of a \( C^1 \)-curve of solutions \( h_k(x, y) = 0 \) in \( S_{f^k, f^{k+1}} \). Let \( H_k \) be the connected component of this curve going through \( (x_k, x_k) \).

Existence of curves with desired properties: We may assume that \( \epsilon > 0 \), the case when \( \epsilon = 0 \) is trivial. If \( \epsilon \in (0, \frac{1}{2}] \), then for any \( a, b \in [-1, 1] \) and any \( (x, y) \in I_{a, b} \), \( P_1 L_c(x, y) \) and \( P_2 L_c(x, y) \) are interior points of the interval \( I_{a, b} \) unless \( x = y = a \) or \( x = y = b \).

By Implicit Function Theorem, the curve \( H_k \) extends to the boundary of the square \( S_{f^k, f^{k+1}} \). Since \( F^*_e(S_{f^k, f^{k+1}}) \subset S_{f^k, f^{k+1}} \), we have that \( P_1 F^*_e(x, y) \) is an interior point of the interval \( I_{f^k, f^{k+1}} \) for every \( (x, y) \in S_{f^k, f^{k+1}} \) unless \( x = y = f^k(0) \) or \( x = y = f^{k+1}(0) \). This implies that the curve \( H_k \) can only meet the boundary on the line segments \( x = f^k(0) \) and \( x = f^{k+1}(0) \). By Proposition 1.1-(2), the curve crosses the diagonal of \( S_{f^k, f^{k+1}} \) once at the point \( (x_k, x_k) \). Therefore, \( H_k \) joins the two line segments.

By symmetry, we have \( G_k \), the connected smooth curve on which \( g_k(x, y) = 0 \). \( G_k \) passes the point \( (x_k, x_k) \) and joins the line segments \( y = f^k(0) \) and \( y = f^{k+1}(0) \) if \( k \in \{0, \cdots, n - 1\} \) and the line segments \( y = 0 \) and \( y = f^2(0) \) if \( k = n \).
Figure 1: The curves $H_0$ and $G_0$ when $n \geq 1$ and the condition (1,2) is satisfied.

To complete the proof of the Theorem, we now show that, under the condition (1,2), the curves $H_k$ and $G_k$ intersect at the point $(x_k,x_k)$ in a way that guarantees an off-diagonal intersection (see an example Figure 1).

**Lemma 5.4** If $(1 - 2\epsilon)^2 f(x_k)^2(x_k) < -1$, then, at the point $(x_k,x_k)$, the slope of the tangent line to $H_k$ is smaller than $-1$ and the slope of the tangent line to $G_k$ is larger than $-1$.

**Proof:** Formulae (5,6) and (5,8) evaluated at $(x_k,x_k)$ give us

$$\frac{\partial h_k}{\partial x}(x_k,x_k) = (f^{2^h})'(x_k)(1-\epsilon,\epsilon) \left(\begin{array}{cc} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{array} \right)^{2^{h-2}} \left(\begin{array}{c} 1-\epsilon \\ \epsilon \end{array} \right),$$

and

$$\frac{\partial h_k}{\partial y}(x_k,x_k) = (f^{2^h})'(x_k)(1-\epsilon,\epsilon) \left(\begin{array}{cc} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{array} \right)^{2^{h-2}} \left(\begin{array}{c} \epsilon \\ 1-\epsilon \end{array} \right) - 1.$$

We have

$$\left(\begin{array}{cc} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{array} \right) = (1-\epsilon) \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) + \epsilon \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),$$

21
and then
\[
\left(1 - \epsilon \begin{array}{c}
\epsilon \\
1 - \epsilon
\end{array} \right)^{2^h} = \sum_{i=0}^{2^h-1} C_{2^h}^{2i}(1 - \epsilon)^{2i} \epsilon^{2^h-2i} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
\[
+ \sum_{i=1}^{2^h-1} C_{2^h}^{2i-1}(1 - \epsilon)^{2i-1} \epsilon^{2^h-2i+1} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

The slope of the tangent lines follow directly from this expression. The slope for \(H_k\) is
\[
S_k := -\frac{\partial h_k(x_k, x_k)}{\partial y(x_k, x_k)}
\]
\[
= -(f^{2^h})'(x_k) \sum_{i=0}^{2^h-1} C_{2^h}^{2i}(1 - \epsilon)^{2i} \epsilon^{2^h-2i} / (f^{2^h})'(x_k) \sum_{i=1}^{2^h-1} C_{2^h}^{2i-1}(1 - \epsilon)^{2i-1} \epsilon^{2^h-2i+1} - 1
\]

The slope for \(G_k\) is the inverse \(1/S_k\). The denominator of \(S_k\) being negative, the condition \(S_k < -1\), which leads to the desired conclusion, is equivalent to \((1 - 2\epsilon)^{2^h} (f^{2^h})'(x_k) < -1\) and the proof is complete. \(\square\)

A Properties of the sequences \(I^n\) and \(J^n\)

The definition of the *-product implies that the infinite sequences
\[
I^n = \{I^n_i\}_{i \in \mathbb{N}} = R^* \ast R^\infty
\]
satisfy the following inductive relations. We have \(I^0 = R^\infty\) and for every \(n \in \mathbb{N}\) and all \(i \in \mathbb{N}\)
\[
I_{2^i+1}^n = R \quad \text{and} \quad I_{2^i+1}^{n+1} = \tilde{I}_i^n,
\]
where \(\tilde{R} = L\) and \(\tilde{L} = R\). (For instance, \(I^1 = (RL)^\infty\), \(I^2 = (RLRR)^\infty\) and \(I^3 = (RLRRRLRL)^\infty\).)

Similarly, the finite sequences \(J^n = \{J_i^n\}_{0 \leq i \leq 2^{n+1}-1} = R^* \ast RC\) satisfy the induction \(J^0 = RC\) and for every \(n \in \mathbb{N}\),
\[
J_{2^i+1}^n = R \quad \text{and} \quad J_{2^i+1}^{n+1} = \tilde{J}_i^n, \quad 0 \leq i \leq 2^{n+1} - 1,
\]
where \(\tilde{C} = C\).

These inductive relations allow us to prove the following properties which are used in the proofs of Proposition 2.2 and Lemma 5.2.

Lemma A.1 For every \(n \in \mathbb{N}\), we have
(A) $I_{2^n-1}^n = R$ (resp. $I_{2^n-1}^n = L$) if $n$ is even (resp. odd),
(B) $\{I_i^n\}_{0 \leq i \leq 2^{n+1}-2} = \{I_i^n\}_{0 \leq i \leq 2^{n+1}-2},$
(C) $\{J_i^n\}_{0 \leq i \leq 2^{n+1}-2} = \{I_i^n\}_{0 \leq i \leq 2^{n+1}-2}$ and $J_{2^{n+1}-1}^n = C,$
(D) the sequence $I^n$ is $2^n$-periodic sequence,
(E) the sequence $\{I_i^n\}_{0 \leq i \leq 2^{n}-1}$ is odd [4].

Proof of the Lemma: (A) We have $I_0^1 = R$ and then the statement holds for $n = 0$. Let $n \geq 0$ be even and assume that $I_{2^n-1}^n = R$. By the relation (A.1), we have $I_{2^{n+1}-1}^{n+1} = I_{2^n-1}^n = L$ and then the statement holds for $n + 1$ which is odd. The proof is similar when $n$ is odd.

(B) For $n = 0$, we have $I_0^1 = R = I_0^0$ and the statement holds. Assume the statement holds for some $n \geq 0$. Then, by the relation (A.1), we have

$$I_{2^{n+1}-1}^{n+2} = I_{2^n-1}^{n+1} = I_i^n = I_{2^{n+1}-1}^{n+1}, \quad 0 \leq i \leq 2^{n+1} - 2,$$

and $I_{2^{n+1}-1}^{n+1} = I_{2^n-1}^n$ for every $i \in \{0, \ldots, 2^{n+1} - 1\}$ which implies that the statement holds for $n + 1$.

The proofs of (C),(D) and (E) are similar and left to the reader. (The proof of (E) is given in [4].)

Proof of Proposition 2.2. (1) Suppose first that the kneading sequence of $f$ is $I^k$ for some $k \in \mathbb{N}$. Then $J_0(f^{2^k}(0)) = I_{2^k-1}^k$ and the result follows from Lemma A.1-(A).

Now if the kneading sequence is $I^n$ for some $n > k$, since we have $2^k - 1 \leq 2^{n+1} - 2$, by Lemma A.1-(B), it follows $J_0(f^{2^k}(0)) = I_{2^k-1}^n = I_{2^k-1}^k$ so the result holds. The same argument and Lemma A.1-(C) show that the result is also true when the kneading sequence is $J^n$ with $n \geq k$. Statement (1) is proved.

(2) The argument is similar to that in the proof of (1). First, suppose that the kneading sequence of $f$ is $I^k$ for some $k \in \mathbb{N}$. We have

$$\{I_i(f^{2^k}(0))\}_{1 \leq i \leq 2^k} = \{I_{i+2^k-1}(1)\}_{1 \leq i \leq 2^k},$$

where the second equality follows from the periodicity of $I^k$ (Lemma A.1-(D)). As before, this property extends to the case where the kneading sequence is either $I^n$ or $J^n$ with $n \geq k$.

(3) The argument is once again similar to that in the proof of (1). When the kneading sequence is $I^k$, the result is given by Lemma A.1-(E). The other cases follow from Lemmas A.1-(B) and A.1-(C).

(4) If the kneading sequence of $f$ is $J^n$, then by Lemma A.1-(C), we have $f^{2^{n+1}}(0) = 0$ and the statement holds (we have equality of sets in this case).
Assume now that the kneading sequence of $f$ is $I^n$ for some $n \geq 1$ (The result in the case where $n = 0$ is obvious). By Statements (2) and (3), one can apply Proposition 2.1-(ii) with $a = f^{2^n}(0)$ and $t = 2^n - 1$ to obtain the inequality $f^{2^n}(0)(f^{2^n+1}(0) - f^{2^n}(0)) < 0$. But the periodicity of $I^n$ implies the inequality $f^{2^n}(0)f^{2^n+1}(0) > 0$ and then the inclusion follows.

**B Proofs of Lemmas 4.1 and 4.2**

**Proof of Lemma 4.1.** We use induction on $t$. Firstly, note that if $(x_1 - x_2, y_1 - y_2) < 0$, then $L_\varepsilon(x_1 - x_2, y_1 - y_2) < 0$. Now, if $(x_1, y_1), (x_2, y_2) \in S_{(0,a)}$ and $(x_1 - x_2, y_1 - y_2) < 0$, then the piecewise monotonicity of $f$ and the monotonicity of $L_\varepsilon$, we have

$$(-1)^s(x_1^1 - x_2^1, y_1^1 - y_2^1) < 0,$$

where $s = 0$ (resp. $s = 1$) if $I_0(a) = L$ (resp. $I_0(a) = R$). The statement holds for $t = 0$.

Now, given $t \in \mathbb{N}$, assume that $\{I_i(a)\}_{1 \leq i \leq t} = \{I_{i-1}(1)\}_{1 \leq i \leq t+1}$ (where all symbols are different from $C$) and suppose that $\{I_i(a)\}_{0 \leq i \leq t+1}$ is even.

If $\{I_i(a)\}_{0 \leq i \leq t}$ is even and $I_{t+1}(a) = I_t(1) = L$, then we have by induction hypothesis $(x_1^{t+1} - x_2^{t+1}, y_1^{t+1} - y_2^{t+1}) < 0$. Using the fact that $f$ is strictly increasing on $[-1,0]$ and the monotonicity of $L_\varepsilon$, we conclude that $(x_1^{t+2} - x_2^{t+2}, y_1^{t+2} - y_2^{t+2}) < 0$. If $\{I_i(a)\}_{0 \leq i \leq t}$ is odd and $I_{t+1}(a) = I_t(1) = R$, we have $(x_2^{t+1} - x_1^{t+1}, y_2^{t+1} - y_1^{t+1}) < 0$ and then $(x_2^{t+2} - x_1^{t+2}, y_2^{t+2} - y_1^{t+2}) < 0$ because now both $f^{t+1}(0)$ and $f^{t+1}(a)$ are positive. The statement then follows for $t + 1$ in the case where $\{I_i(a)\}_{0 \leq i \leq t+1}$ is even. The odd case follows analogously.

**Proof of Lemma 4.2.** We again use induction. If $I_0(a) = L$, then by the piecewise monotonicity of $f$, the assumption on $(x_1, y_1)$ and $(x_2, y_2)$ implies $(f(x_1) - f(x_2), f(y_1) - f(y_2)) < 0$ and then by the definition of $F_\varepsilon$

$$x_1^1 - y_1^1 = (1 - 2\varepsilon)(f(x_1) - f(y_1)) < (1 - 2\varepsilon)(f(x_2) - f(y_2)) = x_2^1 - y_2^1.$$

This implies that either $(-1)^s(x_1^1 - x_2^1, y_1^1 - y_2^1) < 0$ for some $s \in \{0, 1\}$ or $(x_1^1 - x_2^1, y_1^1 - y_2^1) < 0$. Similarly, in the case where $I_0(a) = R$, one obtains that either $(-1)^s(x_1^1 - x_2^1, y_1^1 - y_2^1) < 0$ for some $s \in \{0, 1\}$ or $(x_2^1 - x_1^1, y_2^1 - y_1^1) < 0$. The statement follows for $t = 0$.

Assume now that the statement holds for some $t \in \mathbb{N}$. If the property (i) holds, then as in the proof of Lemma 4.1, one proves that the same property holds for $t + 1$. In the case where the property (ii) holds for $t$, then as for $t = 0$, one proves that either (i) or (ii) holds (or both hold) for $t + 1$. The Lemma is proved.
References


