Let $M^n$ be a compact $C^2$ Riemannian manifold of dimension $n \geq 2$. Let $\text{Diff}^{1+\alpha}(M^n)$ be the space of all $C^{1+\alpha}$ diffeomorphisms of $M^n$ where $0 < \alpha \leq 1$. For a $C^{1+\alpha}$ diffeomorphism $f$ in $\text{Diff}^{1+\alpha}(M^n)$ with a hyperbolic attractor $\Lambda_f$ on which $f$ is topologically transitive, let $U(f)$ be the $C^1$ open set of $\text{Diff}^{1+\alpha}(M^n)$ such that each element in $U(f)$ can be connected to $f$ by finitely many $C^1$ structural stability balls in $\text{Diff}^{1+\alpha}(M^n)$. Then by the structural stability, any element $g$ in $U(f)$ has a hyperbolic attractor $\Lambda_g$ and $g|\Lambda_g$ is topologically conjugate to $f|\Lambda_f$. Therefore, the topological entropy $h(g)$ is a constant function when it is restricted to $U(f)$. However, the metric entropy $h_\mu(g)$ with respect to the SRB measure $\mu = \mu_g$ can vary. We prove that the infimum of the metric entropy $h_\mu(g)$ on $U(f)$ is zero.

1. Introduction and the Main Theorems

Let $M^n$ be a compact $C^2$ Riemannian manifold of dimension $n \geq 2$. Let $\text{Diff}^{1+\alpha}(M^n)$ be the space of all $C^{1+\alpha}$-diffeomorphisms of $M^n$. Suppose $f \in \text{Diff}^{1+\alpha}(M^n)$ has a hyperbolic attractor $\Lambda_f$ on which $f$ is topologically transitive. By the structural stability, there is a sufficiently small $C^1$-neighborhood of $f$, 

$$B^1_\epsilon(f) = \{g \in \text{Diff}^{1+\alpha}(M^n) \mid \|f - g\|_1 < \epsilon\}, \quad \epsilon > 0,$$

where $\|\cdot\|_1$ means the $C^1$ norm, such that $g$ has a hyperbolic attractor $\Lambda_g$ for any $g \in B^1_\epsilon(f)$ and $g|\Lambda_g$ is topologically conjugate to $f|\Lambda_f$. We say $g$ can be connected with $f$ if there are finitely many neighborhoods $\{B^1_\epsilon(f_i)\}_{i=0}^n$ such that $f_0 = g$ and $f_n = f$ and $f_i \in B^1_\epsilon(f_{i+1})$, $0 \leq i \leq n - 1$. Let $U(f)$ be the collection of all diffeomorphisms $g$ in $\text{Diff}^{1+\alpha}(M^n)$ which can be connected with $f$. It is clear that $U(f)$ is an open set of $\text{Diff}^{1+\alpha}(M^n)$ with respect to the $C^1$-topology. Moreover, any map $g$ in $U(f)$ has a hyperbolic attractor $\Lambda_g$ and there exists an SRB measure $\mu_g$ on $\Lambda_g$. Any two maps in $U(f)$ are topologically conjugated by a Hölder continuous homeomorphism. However, this homeomorphism may not necessarily close to the identity. Let $h(g)$ be the topological entropy and let $h_{\mu_g}(g)$ be the metric entropy of $g$ with respect
to its SRB measure $\mu_g$ on $\Lambda_g$. Then $h(g)$ is a constant function when it is restricted to $U(f)$. However, the metric entropy $h_{\mu_g}(g)$ can vary in $U(f)$. It has been shown that the dependence of $\mu_g$ on the map $g$ is smooth when the maps involved have a higher degree of smoothness (see [Ru1] and references therein). In this article, we prove that the infimum of the metric entropy $h_{\mu_g}(g)$ over $U(f)$ is zero.

This result is interesting in several aspects. First, the topological entropy tells the global complexity of a dynamical system. If a dynamical system has a positive topological entropy, it can be thought as a chaotic dynamical system. However, the topological entropy can not tell the level of complexity of a chaotic dynamical system. As a macroscopic quantity, the metric entropy is the standard value that measures the level of complexity of a chaotic dynamical system. Our result says that, given a hyperbolic attractor, there is no barrier to reduce its metric entropy along a $C^1$ homotopic path to a number as small as one wishes, while preserving the uniform hyperbolicity and the topological entropy. Second, our construction of the homotopy gives an example where the connection between a global quantity such as metric entropy and local perturbations can be concretely described. Third, the result leads to many interesting questions one may ask about the nature of the variation of metric entropy within the open neighborhood $U(f)$. For example, is it true that the maximal value of the metric entropy $h_{\mu_g}(g)$ on $U(f)$ is the value of the topological entropy $h(f)$? Is there a way to perturb the diffeomorphism $f$ in a direction so that its metric entropy $h_{\mu_g}(g)$ either decreases or increases monotonically? Are there any local extrema of the metric entropy $h_{\mu_g}(g)$ in $U(f)$?

For the definitions of standard terms such as uniform hyperbolicity, topological conjugacy, and topological and metric entropies, we refer readers to the book [KH]. A map $f \in \text{Diff}^{1+\alpha}(M^n)$ is said to possess a hyperbolic attractor $\Lambda_f$ if there is an open set $\Lambda_f \subset V$ such that $f$ is hyperbolic on $\Lambda_f$ and $\cap_{n=1}^{\infty} f^n(V) = \Lambda_f$.

Suppose $f \in \text{Diff}^{1+\alpha}(M^n)$ has a hyperbolic attractor $\Lambda_f$ on which $f$ is topologically transitive. There are several equivalent definitions of an Sinai-Ruelle-Bowen measure (it is called an SRB-measure) on $\Lambda_f$. We will need the following descriptions:

1. an SRB measure is the unique $f$-invariant measure whose conditional measure restricted on unstable manifolds is absolutely continuous with respect to the induced Lebesgue measure. The density function $\rho(x)$
on a piece of unstable manifold satisfies the condition
\[ \frac{\rho(x)}{\rho(y)} = \prod_{i=1}^{\infty} \frac{\det Df_{x_i}|_{E_{ux_i}^u}}{\det Df_{y_i}|_{E_{uy_i}^u}} = \prod_{i=0}^{\infty} \frac{\det Df_{x_i}^{-1}|_{E_{ux_i}^u}}{\det Df_{y_i}^{-1}|_{E_{uy_i}^u}}, \]
where \( x, y \) are in the same unstable manifold passing certain point, \( x_i = f^{-i}(x), y_i = f^{-i}(y) \) and \( \det Df_{x_i}|_{E_{ux_i}^u}, \det Df_{x_i}^{-1}|_{E_{ux_i}^u} \) denote the Jacobians of \( f \) and \( f^{-1} \) restricted to the unstable subspace at \( x_i \), respectively.

(2) the SRB measure \( \mu = \mu_f \) is the unique invariant measure satisfying the variational principle
\[ h_\mu(f) = \int_\Lambda \log |\det Df_x|_{E_x^u}|d\mu, \]
where \( h_\mu(f) \) is the metric entropy of \( f \) with respect to \( \mu \).

We shall assume that \( f \) is topologically mixing on \( \Lambda_f \). Thus, the SRB measure is unique. But the result can be extended to the general case when \( f \) has several distinct ergodic components using the spectral decomposition theorem \([?]\). The main theorem of the article is the following.

**Theorem A.** Suppose \( f \in \text{Diff}^{\alpha+\alpha}(M^a) \) has a hyperbolic attractor \( \Lambda_f \) on which \( f \) is topologically transitive. Then there is a \( C^1 \) path
\[ H(t) = \{ f_t \in U(f) \mid 0 < t \leq 1 \} \]
such that
\[ \lim_{t \to 0^+} h_{\mu_t}(f_t) = 0. \]
where \( \mu_t \) denotes the unique SRB measure on the hyperbolic attractor \( \Lambda_{f_t} \) for \( f_t \).

To prove this theorem, the idea of perturbing a map at a fixed point in the direction of unstable manifold is quite natural. Since we have a hyperbolic attractor, the metric entropy can be calculated using the formula (1.1). If we reduce the expanding rate near a fixed point, a typical orbit will spend more and more time near the fixed point where \( \log |\det Df_x|_{E_x^u}| \) is small. But the approach by considering such an orbit is difficult because we do not know whether a typical orbit will remain typical after perturbation.

In order to get a control of the SRB measure near the fixed point for perturbed maps, we apply a technique commonly used in studying non-uniformly hyperbolic systems \([HL,H]\). We need to have a distortion estimate of the unstable Jacobian along an orbit independent of the ever-decreasing expansion rate at the fixed point.
For Anosov maps, in the limiting case when $t = 0$, the path $H(t) = \{ f_t \}$ can be so constructed that the limiting map $f_0$ is an almost Anosov map defined initially in [Hu]: it is hyperbolic except for a finite set of $M^n$. For such an almost Anosov map, it was shown that it also has an SRB measure - but the measure is infinite (yet, $\sigma$-finite). We have the following theorem.

Theorem B. Any Anosov diffeomorphism $f$ in $\text{Diff}^{1+\alpha}(M^n)$ with a fixed point $p$ is $C^1$-homotopic to an almost Anosov diffeomorphism $f_0$ that has $\sigma-$finite SRB measure.

Moreover, the SRB measure $\mu_t = \mu_{f_t}$ on $\Lambda_{f_t}$ converges to the SRB measure $\mu_0$ on $\Lambda_{f_0}$ in the following sense.

The metric entropy of a $\sigma$-finite $f_0$-invariant measure $\mu_0$ can be approximated by the metric entropy of the first return map restricted to a subset with a positive measure, i.e.

$$h_{\mu_0}(f_0) = \mu_0(U)h_{\mu_{0,U}}(f_{0,U}),$$

where $U \subset M^n$ is any set with $\mu_0(U) > 0$ and $f_{0,U}$ is the first return map of $U$ and $\mu_{0,U}$ be the induced $f_{0,U}$-invariant measure [?].

Corollary 1. For any subset $U$ of $M^n$ disjoint with some neighborhood of the fixed point $p$, there is an infinite SRB measure $\mu_0$ on $\Lambda_{f_0}$ with $\mu_0(U) = 1$ such that $(\mu_t(U))^{-1}\mu_t$ converges to $\mu_0$ in the weak* sense and

$$(\mu_t(U))^{-1}h_{\mu_t}(f_t) \to h_{\mu_0}(f_0) \quad \text{as} \quad t \to 0^+.$$

The rest of the article is divided into 4 sections. In the next section, we describe in detail how the perturbation is constructed. We show that any $f \in \text{Diff}^{1+\alpha}(M^n)$ with a hyperbolic attractor as its non-wandering set and with a fixed point $p$ can be perturbed successively in an appropriate way within $U(f)$ such that the perturbed map has a desired normal form in a neighborhood of its fixed point. In Section 4, we prove the uniform boundedness of the distortion of the unstable Jacobian. In Section 5, we give the proof of our main theorems.

2. Construction

Suppose $f \in \text{Diff}^{1+\alpha}(M^n)$ has a hyperbolic attractor $\Lambda_f$ on which $f$ is topologically transitive. Therefore, $f|_{\Lambda_f}$ is structural stability. Without loss of generality, we will assume that $f$ has a fixed point $p$. Otherwise, we can consider a periodic orbit. We will construct a family

$$\{ f_t \in U(f) \mid 0 < t \leq 1 \}$$

of maps having hyperbolic attractors $\Lambda_{f_t}$ on which all $f_t$ are topologically transitive. The family satisfies the following main properties:
(1) The fixed point $p$ is preserved for the family and maps in the family are obtained by perturbing $f$ in a small neighborhood of $p$.

(2) The contracting rates along the stable manifolds for maps in the family are bounded above by a constant $0 < \lambda_1 < 1$, while the expanding rates along the unstable manifolds for maps in the family in a small neighborhood of $p$ can be arbitrarily close to 1 as $t \to 0^+$ and the small neighborhood becomes smaller and smaller.

Once this family is constructed such that the distortions (see Section 4) of maps in the family along any unstable manifolds close to the fixed point are bounded, we can estimate, for every map in this family, the density function of the SRB measure near the fixed point $p$. Furthermore, we will prove that the infimum of metric entropies with respect to SRB measures of maps in this family is zero.

2.1. Linearizing the map near $p$. We first show the following lemma

\textbf{Lemma 1.} For the given map $f$ as above, there is another $g \in \text{Diff}^{1+\alpha}(M^n)$ having the same fixed point $p$ such that the following properties hold:

(1) $g \in U(f)$ and $C^1$ close to $f$.

(2) $g$ is identical to $f$ outside an $\epsilon_1$-neighborhood of $p$.

(3) There is a coordinate system $\varphi : \mathbb{R}^n \to M^n$ near $p$ such that $\varphi(0) = p$ and $\varphi^{-1} \circ f \circ \varphi$ is a linear map in an $\epsilon_0(\epsilon_1)$-open ball centered at $0 \in \mathbb{R}^n$ of the form

\begin{equation}
L \begin{pmatrix} x^u \\ x^s \end{pmatrix} = \begin{pmatrix} Ax^u \\ Bx^s \end{pmatrix},
\end{equation}

where $x = (x^u, x^s)$ are coordinates provided by the unstable and stable subspaces of $Df$ at $p$.

Let $n^u$ and $n^s$ be the dimensions of the unstable and stable subspaces of $Df$ at $p$. We may assume that the coordinate system provided by the unstable and stable subspaces are just Euclidean spaces $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$ with $\mathbb{R}^{n^u} \otimes \mathbb{R}^{n^s} = \mathbb{R}^n$.

\textit{Proof.} We identify $p$ with the origin 0 of $\mathbb{R}^n$. There exists an $\epsilon_1$-neighborhood of $p$ in $M^n$ such that $f$ is $C^1$ close to its linear approximation $Df(0)$ in this $\epsilon_1$-neighborhood. Identify this $\epsilon_1$-neighborhood with an open ball $B_{\epsilon_1}(0)$ of radius $\epsilon_1$ centered at 0 in $\mathbb{R}^n$.

In the coordinate system given by the unstable and stable subspaces $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$, the linear operator $Df(0)$ takes the form

$$Df(0) \begin{pmatrix} x^u \\ x^s \end{pmatrix} = \begin{pmatrix} Ax^u \\ Bx^s \end{pmatrix},$$
where \( A \) is an expanding linear map and \( B \) is an contracting linear map.

Take a smaller open ball \( B_{\epsilon_0}(0) \) of radius \( 0 < \epsilon_0 < \epsilon_1 \) centered at 0. Extend the map \( (f - Df(0))|_{B_{\epsilon_0}(0)} \) to a \( C^{1+\alpha} \) map \( f_1 \) on \( B_{\epsilon_1}(0) \) so that \( \|f_1\| + \|Df_1\| \) is small. This can be done as long as the ratio \( \epsilon_0/\epsilon_1 \) small enough. Let

\[
D(\epsilon_0) = \sup_{x \in B_{\epsilon_0}(0)} \{\|f(x) - Df(0)x\| + \|Df(x) - Df(0)x\|\}.
\]

We can require that

\[
\|f_1\| + \|Df_1\| \leq 2D(\epsilon_0).
\]

Clearly,

\[
\lim_{\epsilon_0 \to 0} D(\epsilon_0) = 0.
\]

Let \( \tau(x) \) be a smooth map such that it is the identity inside \( B_{\epsilon_0}(0) \) and it maps every point to zero outside of \( B_{\epsilon_1}(0) \). Define

\[
g = \begin{cases} 
  f & \text{if } x \notin B_{\epsilon_1}(0) \\
  f - \tau \cdot f_1 & \text{if } x \in B_{\epsilon_1}(0).
\end{cases}
\]

The map \( g \) is linear in \( B_{\epsilon_0}(0) \). The \( C^1 \)-distance between \( g \) and \( f \) is bounded by the \( \|f_1\| + \|D\tau\| \|Df_1\| \). Note that \( \|D\tau\| \), depending on the ratio \( \epsilon_0/\epsilon_1 \), can be made as smooth as we wish with derivatives uniformly bounded as long as \( \epsilon_0 \leq \epsilon_1/2 \). So, we can select \( \epsilon_1 \) and \( \epsilon_0 \) with \( \epsilon_0/\epsilon_1 \) sufficiently small such that the \( C^1 \)-distance between \( g \) and \( f \) is less than any given number \( \delta_0 \). It implies that \( g \in \text{Diff}^{1+\alpha}(M^n) \) is in a \( C^1 \)-neighborhood of \( f \). Thus, \( g \) has a hyperbolic attractor \( \Lambda_g \) on which \( g \) is topologically transitive. This completes the proof.

Note that, other than using \( Df(0)x \), the linear approximation, we can use other nonlinear choices near 0. The only condition of these choices needs to be satisfied is that these choices are \( C^{1+\alpha} \) and sufficiently \( C^1 \)-close to \( f \) in an \( \epsilon_1 \)-neighborhood of 0.

**Corollary 2.** The linear map in (2.1) near the origin 0 can be changed to any \( C^{1+\alpha} \) map which is \( C^1 \) close to \( f \) in a neighborhood of 0, in particular, a map in the form of

\[
x = \begin{pmatrix} x^u \\ x^s \end{pmatrix} \to \begin{pmatrix} (A + C(x^s))x^u \\ Bx^s \end{pmatrix},
\]

where \( C(x^s) \) is a \( C^{1+\alpha} \) map with \( C(0) = 0 \).

Now we further perturb the map \( g \) inside the small neighborhood \( B_{\epsilon_1}(0) \). The perturbation preserves the form (2.2) near the point 0. We need to perturb the map along a homotopy path in \( U(f) \) so that the matrices \( A \) and \( B \) become
diagonal matrices first. In the unstable direction, we need to further reduce the expanding rates to a number \(1 + \delta\) with \(\delta\) small. In the stable direction, the contracting rate will be a constant. The perturbation is no longer in a small \(C^1\) neighborhood of \(f\). But we will show that the perturbed maps are still uniformly hyperbolic and homotopic to \(f\) in \(U(f)\). For simplicity, we give the proof in the case when \(C(x^s)\) is zero. In the case \(C(x^s)\) is not zero, we apply Corollary 2 to Lemma 1 one more time since \(C(x^s)\) is small when the norm \(|x^s|\) is small.

**Lemma 2.** Let \(g\) be the map obtained in Lemma 1. Assume that a \(C^{1+\alpha}\) diffeomorphism \(g_1 \in \text{Diff}^{1+\alpha}(M^n)\) has the following properties:

1. \(g_1(x) = f(x)\) for \(x \not\in B_{\epsilon_0}(0)\) and \(g_1(0) = 0\).
2. \(g_1(x)\) preserves the direct product structure \(\mathbb{R}^n_u\) and \(\mathbb{R}^n_s\) for \(x \in B_{\epsilon_0}(0)\), i.e., \(g_1(x)\) takes the form

\[
g_1 \left( \begin{array}{c} x^u \\ x^s \end{array} \right) = \left( \begin{array}{c} \tilde{A}(x^u) \\ \tilde{B}(x^s) \end{array} \right),
\]

with \(\tilde{A}(x^u) \in \mathbb{R}^n_u\) and \(\tilde{B}(x^s) \in \mathbb{R}^n_s\) for

\[
x = \left( \begin{array}{c} x^u \\ x^s \end{array} \right) \in B_{\epsilon_0}(0).
\]

3. \(\tilde{A}(x^u)\) is expanding on \(\mathbb{R}^n_u\) and \(\tilde{B}(x^s)\) contracting on \(\mathbb{R}^n_s\), both uniformly.

Then, when \(\epsilon_0\) is sufficiently small, \(g_1\) has a hyperbolic attractor \(\Lambda_g\).

A direct corollary of this lemma is that when \(\tilde{A}(x^u)\) and \(\tilde{B}(x^s)\) are homotopic to the linear maps \(Ax^u\) and \(Bx^s\) in \(\text{Diff}^{1+\alpha}(M^n)\), then \(g_1\) is in \(U(f)\).

**Proof.** Since we assume that \(f\) has a hyperbolic attractor \(\Lambda_f\) on which \(f\) is topologically transitive, so is \(g\) in Lemma 1. Let \(\Lambda_g\) be the corresponding hyperbolic attractor of \(g\). Take an \(\epsilon\)-neighborhood \(O_\epsilon(\Lambda_g)\) such that the stable and unstable subspaces are extended to the entire neighborhood of the hyperbolic set \(\Lambda_g\) [KH]. We may also assume that \(\epsilon_0 \leq \epsilon\), thus the entire \(B_{\epsilon_0}(0)\) is contained in \(O_\epsilon(\Lambda_g)\). Take an appropriate coordinate system (almost Lyapunov metric) on the neighborhood \(O_\epsilon(\Lambda_g)\) such that the stable and unstable subspaces are nearly orthogonal. At each point \(x\) inside the set \(B_{\epsilon_0}(0)\), the unstable and stable subspaces \(E^u_q\) and \(E^s_q\) for \(Dg(q)\) are within \(C\epsilon_0\)-distance (Grassmannian distance) of \(\mathbb{R}^n_u\) and \(\mathbb{R}^n_s\), respectively, for some constant \(C\). We assume that for some \(\lambda < 1 < \mu\), we have

\[
\|Dg(x)v\| \geq \mu\|v\|, \quad v \in E^u_x, \quad \|Dg(x)w\| \leq \lambda\|w\|, \quad w \in E^s_x.
\]
To show that $g_1$ is uniformly hyperbolic, we first prove that both $Dg_1$ and $D^{-1}g_1$ admit an invariant cone field and they induce contracting operators on these cone fields. As a consequence, we obtain exponential splitting along any orbit of $g_1$ inside the open neighborhood $O_\varepsilon(\Lambda_g)$. We then show the expanding and contracting rates along the invariant subspaces of the exponential splitting are uniform. In the process of the proof, we may need to further decrease the radius $\varepsilon_0$ for $B_\varepsilon(0)$. But such modification will not affect the validity of the arguments since it depends only on the map $g$.

The cone field $C^u_x$ along the unstable subspaces is defined as follows.

$$C^u_x = \begin{cases} (v, w) \in T_xM^n : v \in E^u_x, w \in E^s_x, \|w\| \leq \alpha\|v\| & \text{for } x \notin B_{\varepsilon_0}(0) \\ (v, w) \in T_xM^n : v \in R^{n^u}, w \in R^{n^s}, \|w\| \leq \alpha\|v\| & \text{for } x \in B_{\varepsilon_0}(0) \end{cases},$$

where $0 < \alpha < 1$. The cone field $C^s_x$ along the stable subspaces is defined in the similar way. Note that this cone field is not a continuous one. But this should not affect the Hölder continuity of stable and unstable subspaces because of the invariance.

We just need to show that the derivative operator is contracting on this cone field, since the invariance follows the contraction. Given a point $x$, if both $x$ and $g_1(x)$ are inside or outside of $B_{\varepsilon_0}(0)$, the contraction follows automatically since $E^u$ and $E^s$ are invariant under $Dg_1$ when it is restricted on the outside of $B_{\varepsilon_0}(0)$ and since $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$ are invariant under $Dg_1$ when it is restricted on the inside of $B_{\varepsilon_0}(0)$. So, we just need to verify two situations: Case 1, $x \in B_{\varepsilon_0}(0)$ but $g_1(x) \notin B_{\varepsilon_0}(0)$; Case 2, $g_1(x) \in B_{\varepsilon_0}(0)$ but $x \notin B_{\varepsilon_0}(0)$.

**Case 1:** $x \in B_{\varepsilon_0}(0)$ but $g_1(x) \notin B_{\varepsilon_0}(0)$.

Take $(v, w) \in C^u_x$, $v \in E^u_x$, $w \in E^s_x$. Then $Dg_1(x)(v, w) = (D\bar{A}v, D\bar{B}w)$ is in $\mathbb{R}^{n^u} \oplus \mathbb{R}^{n^s}$ coordinate system. In the $E^u \oplus E^s$ coordinate system, we have

$$(P_{11}D\bar{A}v + P_{12}D\bar{B}w, P_{21}D\bar{A}v + P_{22}D\bar{B}w) \in E^u \oplus E^s,$$

where $P_{ij}$ denotes the coordinate change matrices and $D\bar{A}, D\bar{B}$ are derivative operators of $\bar{A}(x^u)$ and $\bar{B}(x^s)$. Since $p$ is a fixed point of $g$ and $g_1$ is $C^0$-close to $g$, $\|g_1(x)\| \leq 2\varepsilon_0$. So, we have $\|P_{11}\|, \|P_{22}\| \geq (1 - \varepsilon)$ and $\|P_{12}\|, \|P_{21}\| \leq \varepsilon$ where $\varepsilon = C\varepsilon_0$ for some constant $C$. Estimating the norms, we have

$$\|P_{11}D\bar{A}v + P_{12}D\bar{B}w\| \geq ((1 - \varepsilon)\mu - \alpha\lambda\varepsilon)\|v\|;$$

$$\|P_{21}D\bar{A}v + P_{22}D\bar{B}w\| \leq (\varepsilon\bar{\mu} + \alpha\lambda(1 - \varepsilon))\|v\|,$$

where $\bar{\mu}, \bar{\lambda}$ denote the maximal expanding rate and minimal contracting rate of $\bar{A}$ and $\bar{B}$, respectively. we have

$$\|P_{21}D\bar{A}v + P_{22}D\bar{B}w\| \leq \bar{\alpha} \left( \frac{\lambda(1 - \varepsilon) + \varepsilon\bar{\mu}/\alpha}{(1 - \varepsilon)\mu - \alpha\lambda\varepsilon} \right) \|P_{11}D\bar{A}v + P_{12}D\bar{B}w\|. \quad (2.3)$$
When $\epsilon_0$ is adequately small, depending on the ratio of contracting to expanding rates and $\alpha$, we have

$$\frac{\lambda(1-\epsilon) + \epsilon\tilde{\mu}/\alpha}{(1-\epsilon)\mu - \alpha\epsilon \lambda} < 1.$$ 

The proof in Case 2 is completely parallel. Thus we proved the invariance of the cone field around the stable subspaces.

We now proceed to prove the uniform hyperbolicity of $g_1$ along any orbit in $O_{\epsilon}(\Lambda_g)$. We consider only the unstable case. The stable case is the same. Given any orbit segment of length $m$, $x_1, x_2, x_3, \cdots, x_m, x_{i+1} = f(x_i)$, since $p$ is a fixed point, we can find another smaller number $\epsilon_2 \leq \epsilon_0$ neighborhood around $p$ such that for any consecutive pairs of points in the orbit segment $x_i, x_{i+1}, i = 1, \cdots, m - 1$, we have either both points are in $B_{\epsilon_0}(0)$ or both are outside of $B_{\epsilon_2}(0)$. Note that the estimation obtained in (2.3) remain valid if we replace $\epsilon_0$ by a smaller number $\epsilon_2$. Thus, when both $x$ and $g_1(x)$ are outside of $B_{\epsilon_2}(0)$, take $(v, w) \in C_u x, \|w\| = \tilde{\alpha}\|v\|$ for some $0 \leq \tilde{\alpha} < \alpha < 1$, using the invariant exponential splitting of $g$, we have

$$\|Dg_1(x)(v, w)\| = \|(Dg_1(x)v, Dg_1(q)w)\| = \max(\|Dg_1(x)v\|, \|Dg_1(x)w\|) = \|Dg_1(x)v\| \geq \mu\|v\|,$$

where we are using an equivalent Finsler metric in estimating the expanding rate. When both $x$ and $g_1(x)$ are inside of $B_{\epsilon_2}(0)$, in the $R^n_{u}$ and $R^n_{s}$ coordinate, we have the same estimation in Finsler metric. We thus conclude that $g_1$ is uniformly hyperbolic along any orbit inside $O_{\epsilon}(\Lambda_g)$. $\square$

**Corollary 3.** The conclusion of Lemma 2 remains valid if we replace $Ax^u$ with $Ax^u + C(x^s)x^u$ of Corollary 2.

Lemma 2 gives us plenty of freedom to perturb the map $f$ inside a fixed neighborhood $B_{\epsilon_0}(0)$. In order to prove a bounded distortion lemma along unstable manifold for a family of diffeomorphism of the form $g_1$ satisfying conditions in Lemma 2, we will need special forms of maps $\tilde{A}(x^u)$ and $C(x^s)$ near the fixed point $p$.

2.2. **The map $f_t$.** We now construct a family

$$H(t) = \{f_t \in U(f) \mid 0 \leq t \leq 1\}$$

such that each map $f_t$ for $0 < t \leq 1$ in this family satisfies the conditions in Lemma 2 and the infimum of metric entropies of maps in this family with respect to SRB measures is zero.

Recall that $n^s$ and $n^u$ are dimensions of $\mathbb{R}^{n^s}$ and $\mathbb{R}^{n^u}$, respectively. Take an even number $m : n^u < m \leq n^u + 2$, and denote $\beta = 1/m$. Hence, $\beta n^u < 1$. 

INFIMUM OF METRIC ENTROPY
Take a family of smooth increasing functions $\phi_t : [0, 1/2) \to \mathbb{R}^+$ that satisfies the following conditions.

1. $\phi_t(r) = (1 - r^m)^{-\beta}$ for $r \geq t$, 
2. $\phi_t(r) = (1 - (t/2)^m)^{-\beta} > 1$ is a constant for $0 \leq r \leq t/2$, 
3. $\phi''_t(r)$ is bounded uniformly in $t$ and nonnegative when $t^2 \leq r \leq t$.

Note that if $r = 1/n^\beta \in [t, 1/2)$, then $\phi'(r) > 0$ and 

$$\frac{1}{n^\beta} \phi_t(\frac{1}{n^\beta}) = \frac{1}{(n-1)^\beta}.$$ 

Another property that we need in the proof of the bounded distortion lemma is that $\phi_t(r)$ satisfies the following inequality.

$$\phi_t(r) \geq 1 + C r^\gamma$$

where $C > 0, \gamma > 0$ are constants independent of $t$. Indeed, when $r > 0$ is small,

$$\phi_t(r) \geq 1 + \beta r^m + \text{higher order terms}.$$ 

Take two open neighborhoods $\Omega_0 \subset \Omega_1$, we may choose open balls, such that $p \in \Omega_0 \subset \overline{\Omega}_0 \subset \Omega_1$. Denote $\kappa^s = \|Df(0)|_{E^s_0}\|$, the contracting rate at 0 in the stable direction. Define a family of diffeomorphisms $f_t$ such that

1. $f_t(x^u, x^s) = \left(\phi_t(\|x^u\|) + \|x^s\|^2 x^u, \kappa^s x^s \right)$, $x = (x^u, x^s) \in \Omega_0$;
2. $f_t(x) = f(x), x \notin \Omega_1$;
3. $f_t$ is uniformly hyperbolic for all $t > 0$ and the contracting rate is independent of $t$.
4. The expanding rate outside of $\Omega_0$ is bounded below by a constant $\mu > 1$ with $\mu$ independent of $t$.

We also require that the second derivative of $f_t$ is uniformly bounded for all $t$ and $x \in M^n$. The existence of $f_t$ is guaranteed by Lemma 1 and Lemma 2 when $t$ is sufficiently small. We note that $f_t$ is $C^\infty$ on $\Omega_0$ since $m$ is an even number. Since $f_t|\Omega_0$ is homotopic to $f|\Omega_0$, we have that $f_t \in U(f)$ for all $0 < t \leq 1$ by the statement after Lemma 2. The hyperbolic attractor of $f_t$ is denoted by $\Lambda_t$.

3. Preliminaries

We list further properties of the maps constructed in the previous section with only sketches of proofs since details can be found in many books such as [KH, HPS]. Let $E^s_z(f_t)$ and $E^u_z(f_t)$ be the stable and unstable manifold with respect to $f_t$. 


Lemma 3.1. The maps $x \to \{ E^u_x(f_t) \}$ and $x \to \{ E^s_x(f_t) \}$ are Hölder continuous and the Hölder exponents and constants can be chosen in a way independent of $0 < t \leq 1$. More precisely, there exist $\theta > 0$, $H > 0$ such that for all $t > 0$, $x, x' \in M^n$,

$$d(E^u_x(f_t), E^u_{x'}(f_t)), d(E^s_x(f_t), E^s_{x'}(f_t)) \leq H \|x - x'\|^\theta,$$

where the distance between subspaces is the Grassmannian distance.

Proof. Note that $\|Df_t(x)|E^u_x(f_t)\| \geq 1$ and $\|Df_t(x)|E^s_x(f_t)\| \leq \lambda^s < 1$ for all $x \in \Lambda_t$ and $t \geq 0$. The conclusions follow from the fact that the Hölder exponent and constant depend only on the Lipschitz constant of the map $f_t$ and the gap between the expansion and contraction rates. See [HPS]. \hfill \Box

For $\varepsilon > 0$, we denote

$$E^u_{x,f_t}(\varepsilon) = \{ v \in E^u_x(f_t) : |v| \leq \varepsilon \} \quad \text{and} \quad E^s_{x,f_t}(\varepsilon) = \{ v \in E^s_x(f_t) : |v| \leq \varepsilon \}$$

and

$$E_{x,f_t}(\varepsilon) = E^u_{x,f_t}(\varepsilon) \times E^s_{x,f_t}(\varepsilon).$$

Proposition 3.2. For each $t \geq 0$, there exist two continuous foliations $\mathcal{F}^u(f_t)$ and $\mathcal{F}^s(f_t)$ on $\Lambda_t$ tangent to $E^u_x(f_t)$ and $E^s_x(f_t)$ respectively for which the following statements hold.

1. The leaf of $\mathcal{F}^s(f_t)$ through $x$, denoted by $\mathcal{F}^s(x, f_t)$, is the stable manifold at $x$, i.e.

$$\mathcal{F}^s(x, f_t) = W^s(x, f_t)$$

$$\{ x' \in \Lambda_t : \exists C = C'_x, \ s.t. \ d(f^n_t(x), f^n_t(x')) \leq C(n_t^s)^n \forall n \geq 0 \},$$

where $n_t^s$ denotes the contracting rate of $f_t$ on the stable manifold.

2. The leaf of $\mathcal{F}^u(f_t)$ through $x$, denoted by $\mathcal{F}^u(x, f_t)$, is the unstable manifold

$$\mathcal{F}^u(x, f_t) = W^s(x, f_t)$$

$$\{ x' \in \Lambda_t : \exists C = C'_x, \ s.t. \ \lim_{n \to \infty} d(f^{-n}_t(x), f^{-n}_t(x')) = 0 \}.$$

3. There exist constants $\delta > 0$ and $D > 0$ such that for all $t \geq 0$, $x \in \Lambda_t$, if $\mathcal{F}^u_{x,f_t}$ is the component of $\mathcal{F}^u(x, f_t) \cap \text{exp}_x E^s_{x,f_t}(\delta)$ containing $x$, then $\exp_x^{-1} \mathcal{F}^u_{x,f_t}(x, f_t)$ is the graph of a function

$$\phi^u_{x,f_t} : E^u_{x,f_t}(\delta) \to E^s_{x,f_t}(\delta)$$

with $\phi^u_{x,f_t}(0) = 0$ and $\|\phi^u_{x,f_t}\|_{C^{1+\alpha}} \leq D$. The analogous statement holds for $\mathcal{F}^s_{x,f_t}(x, f_t)$. 
For convenience we will write \(W^u(x, f_i) = \mathcal{F}^u(x, f_i), W^s_\delta(x, f_i) = \mathcal{F}^s_\delta(x, f_i),\) etc. and refer to \(W^u(x, f_t)\) and \(W^s_\delta(x, f_t)\) as the “unstable manifold” and “local unstable manifold”, respectively, at \(x\).

Further, for simplicity, we use \(E^u, E^s, W^u\) and \(W^s\) to stand for invariant subspaces and sub-manifolds, when doing so will not cause much confusion.

For \(x' \in W^s(x, f_t)\), let \(d^s(x, x')\) denote the distance between \(x\) and \(x'\) measured along \(W^s(x, f_t)\) with the induced metric. For \(x'' \in W^u(x, f_t)\), the distance \(d^u(x, x'')\) is defined similarly.

**Lemma 3.3.** There is \(J^s > 0\) such that for any \(x \in \Lambda_t, x' \in W^s(x, f_t),\)
\[
\frac{|\det Df_t^s(x)|E^s(f_t)|}{|\det Df_t^u(x')|E^u(f_t)|} \leq J^s(d^s(x, x'))^\theta, \quad \forall t > 0.
\]

**Proof.** It follows from the standard arguments and the fact that \(f_t\) is uniformly contracting along the stable manifold and \(E^u(f_t)\) is Hölder continuous. \(\square\)

One of the important ingredients in the proof of the main theorem is that the \(W^s\)-foliation for \(f_t\) is absolute continuous with a uniformly bounded Jacobian.

Let \(\Delta_1\) and \(\Delta_2\) be two \(W^u\)-leaves for \(f_t\), a holonomy map \(\theta : \Delta_1 \to \Delta_2\) is defined by sliding along the \(W^s\)-leaves for \(f_t\), i.e. for \(x \in \Delta_1, \theta(x) \in \Delta_2 \cap W^s(x, f_t)\).

For \(x' \in W^s(x, f_t)\), let \(d^s(x, x')\) denote the distance between \(x\) and \(x'\) measured along \(W^s(x, f_t)\), and for \(x'' \in W^u(x, f_t)\), let \(d^u(x, x'')\) be defined similarly. We have the following proposition.

**Proposition 3.4.** Given \(D_1 > 0\), there exists \(J^s_1 > 0\) such that for every \((\Delta_1, \Delta_2, \theta)\) with \(d^s(\theta(x), x) < D_1\), for \(x \in \Delta_1, \text{ for every } x' \in \Delta_1, \varepsilon > 0\) with \(B^u(x', \varepsilon) \in \Delta_1,\)
\[
m^u(B^u(x', \varepsilon)) \leq J^s_1 m^u(\theta B^u(x', \varepsilon)).
\]

**Proof.** Let \(x_1 \in \Delta_1\), and let \(\mathcal{D}\) be any small disk in \(\Delta_1\) containing \(x_1\). We will argue that \(m^u(\Delta_1) \approx m^u(\theta \Delta_1)\), where \(m^u\) denotes the Lebesgue measure on \(W^u\)-sub-manifolds for \(f_t\), and “\(\approx\)” means “up to a constant”.

Let
\[
\kappa^s_+ = \kappa^s_+(f_t) = \max\{\|Df_t(x)|E^s_\delta(f_t)\| : x \in \Lambda_t\}
\]
and
\[
\kappa^s_- = \kappa^s_-(f_t) = \max\{\|Df_t^{-1}(x)|E^s_\delta\|^{-1} : x \in \Lambda_t\},
\]
that is, \(\kappa^s_+\) and \(\kappa^s_-\) are the norm and minimal norm of \(Df_t\) restricted to the stable bundle \(E^s(f_t)\), respectively.
By taking a sufficiently large iterate of $f_t$, we may assume that $f^n_t(D)$ and $f^n_t(\theta D)$ are close sufficiently so that the “diameters” of $f^n_t(D)$ and $f^n_t(\theta D)$ are much larger than the distance $d^s(x', \theta x')$ for any $x' \in D$.

Take a finite cover $\{B^u(x'_i, r)\}_{i=1}^k$ of $f^n_t(D)$ consisting balls of radius $r$, where $r = 3(\kappa^u)^n$ such that for any $x'' \in f^n_t(D)$, there are at most $C_1$ such balls covering this point, where $C_1$ only depends on the dimension of $W^u(x, f_t)$.

Here we also use $\theta$ denote the holonomy map from $f^n_t(\Delta_1)$ to $f^n_t(\Delta_2)$. Since $d^s(x'', \theta x'') \leq (\kappa^u)^n$ for any $x'' \in f^n_t(D)$, we know that $\theta B^u(x'_i, r)$ contains a ball of radius at least $(\kappa^u)^n$ and is contained in a ball of radius at most $5(\kappa^u)^n$.

Therefore, we have

\[(3.1)\quad m^u(B^u(x'_i, r)) \approx m^u(\theta B^u(x'_i, r)).\]

By Lemma 3.3, we have

\[(3.2)\quad \left| \det Df^n_t(x'')|_{E^u_{x''}(f_t)} \right| \approx \left| \det Df^n_t(\theta x'')|_{E^u_{\theta x''}(f_t)} \right| \quad \forall x'' \in f^n_t(B^u(x'_i, r)).\]

Using the fact that the $C^2$ norm of $f_t$ is uniformly bounded, $W^u(x, f_t)$ are $C^{1+\alpha}$ leaves, and $f_t$ is is expanding on $W^u(x, f_t)$, we have for all $x''_1, x''_2 \in f^n_t(B^u(x'_i, r))$

\[(3.3)\quad \frac{\left| \det Df^n_t(x''_1)|_{E^u_{x''_1}(f_t)} \right|}{\left| \det Df^n_t(x''_2)|_{E^u_{x''_2}(f_t)} \right|} \approx \prod_{j=0}^{n-1} \left( 1 \pm \text{const} \cdot d^u(f^n_t(x''_1), f^n_t(x''_2)) \right) \leq \left( 1 \pm \text{const} \cdot \text{diam } B^u(x'_i, r) \right)^n \approx \left( 1 \pm \text{const} \cdot D_1(3\kappa^u)^n \right)^n \approx \text{const}.\]

Combining (3.1)-(3.3), we get

\[(3.4)\quad m^u(f^n_t(B^u(x'_i, r))) \approx m^u(f^n_t(\theta B^u(x'_i, r))).\]

Let $S$ be the union of $f^n_t(B^u(x'_i, r))$ that belong to $D$. Since

\[\text{diam}(f^n_t(B^u(x'_i, r))), \text{diam}(\theta(f^n_t(B^u(x'_i, r)))) \to 0 \quad \text{as} \quad n \to \infty,
\]

it follows that

\[m^u(S) \approx m^u(D) \quad \text{and} \quad m^u(\theta S) \approx m^u(\theta D).
\]

Also recall that each point in $D$ belong to at most $C_1$ sets of the form $f^n_t(B^u(x'_i, r)$, then (3.4) implies $m^u(D) \approx m^u(\theta D)$.

\[\square\]
4. Distortion Estimates

We now estimate the distortion of the Jacobian along the unstable direction. We use the $\mathbb{R}^n \oplus \mathbb{R}^{n^*}$-coordinate near the fixed point 0 and use Euclidean metric for convenience. Let

$$S = S_\delta = \{ x = (x^u, x^s) \in \Omega_1 : |x^u|, |x^s| \leq \delta \}.$$ 

Denote $S^+ = f_t(S) \setminus S$. It is clear that $S^+$ is independent of $t$ for all small $t$.

We denote $y = (y^u, y^s) = (y^u_0, y^s_0)$ and $z = (z^u, z^s) = (z^u_0, z^s_0)$. We also denote $f_t^{-i}y = y_i = (y^u_i, y^s_i)$ and $f_t^{-1}z = z_i = (z^u_i, z^s_i)$.

**Lemma 4.1.** Let $\delta > 0$ be given sufficiently small. For any $x \in S^+ \setminus W^u_{\delta^+}(0, f_t)$, if

$$d(x, W^u_{\delta^+}(0, f_t)) \to 0,$$

then

$$d(f_t^{-n}(x), W^s_{\delta}(0, f_t)) \to 0,$$

and the convergence is uniform for all $x \in S^+$ and $t \in I$, where $n = n(x)$ the largest integer such that $f_t^{-1}x, \ldots, f_t^{-n}x \in S$.

**Proof.** First, we note that for any $z \in S^+ \cap W^u_{\delta^+}(0, f_t)$, $f_t^{-j}z \to 0$ uniformly with $z$ and $t$ as $j \to \infty$, since this is true for $t = 0$ and $f_t^{-j}z$ converges to 0 faster than that $f_0^{-j}z$.

To consider the case $x \in S^+ \setminus W^u_{\delta^+}(0, f_t)$, we claim that for any $\hat{y}$ and $\hat{z}$, $|\hat{y}^s| > |\hat{z}^s|$ and $|\hat{y}^u| \leq |\hat{z}^u|$ imply $|\hat{y}^u_i| \leq |\hat{z}^u_i|$. Otherwise, there would be a point $\hat{x}$ in a curve joining $\hat{y}$ and $\hat{z}$ near which $|(f_t^{-1}\hat{x})^u|$ increases with $|\hat{x}^s|$. It contradicts the fact that $|(f_t^{-1}x)^u|$ decreases with $|x^s|$ for $x = (x^u, x^s)$.

This claim implies inductively that for $y = (x^u, x^s)$ and $z = (x^u, 0)$, we always have $|y^u_i| \leq |z^u_i|$ for all $0 \leq i \leq n$. So if $x \in S^+ \setminus W^u_{\delta^+}(0, f_t)$, we have $d(x_n, W^s_{\delta^+}(0, f_t)) \leq d(z_n, W^s_{\delta^+}(0, f_t))$. Note that $f_t^{-1}$ is expanding at a constant rate in the stable direction, we get $n = n(x) \to \infty$ uniformly as $d(x, W^u_{\delta^+}(0, f_t)) \to 0$. Hence the result of the lemma follows. 

**Lemma 4.2.** Given any $\delta > 0$, there exist $J_0 \geq 1$ independent of $t$ such that for any $x \in S^+$, $y, z \in W^u(x, S^+)$ for $f_t$,

$$\log \left| \frac{\det Df_t^{-n}(z)|_{E^u_t(f_t)}}{\det Df_t^{-n}(y)|_{E^u_t(f_t)}} \right| \leq J_0 d(y, z)^{1/r},$$

whenever $f_t^{-1}x, \ldots, f_t^{-n}x \in S$. 

Proof. We assume that $\delta$ is small enough such that $S^+ \subset \Omega_0$. So we can use (2.4) for the map $f_t$.

Let $\delta^+ = \delta \phi_i(\delta)$, the outer radius of $S^+$.

We only need to consider the case that $t$ is small since $f_t$ is uniformly hyperbolic for all $t$ away from 0.

We also need to consider only the case when $x$ is sufficiently close to $W^u_{\delta^+}(0, f_t)$, since otherwise the time that the backward orbit of $x$ leaves the set $S$ is bounded, and the inequality (4.1) surely holds. Further, we may assume that $n$ is the largest integer such that $f_t^{-n}x, \ldots, f_t^{-1}x \in S$.

By (2.4), we have

$$Df_t(x)|_{R^n} = (\phi_t(|x^u|) + |x^s|^2) I + x^u \cdot \phi_t'(|x^u|)|x^u|^{-1}(x^u)^T,$$

where $I$ is the $n^u \times n^u$ identity matrix, and $(x^u)^T$ is the transpose of the column vector $x^u$. Using the fact that $\det(aI + bxT) = a^{n-1}(a + b|x|^2)$ for an $u^n \times u^n$ matrix $aI + bxT$, we have

$$\det Df_t(x)|_{R^n} = (\phi_t(|x^u|) + |x^s|^2)^{n-1}(\phi_t(|x^u|) + |x^s|^2 + \phi_t'(|x^u|)|x^u|))$$

It means that $\det Df_t(x)|_{R^n}$ only depends on the norms $|x^u|$ and $|x^s|$.

Case one: $y^u, z^u$ and $p^u = 0^u$ are on the same line: $z^u = ay^u$ for some $0 < a < 1$.

Note that $|z^s_n|$ is close to $\delta$ by the choice of $n$. So $|z^s_n| - |z^s_{n-1}|$ is close to $(1 - \kappa^n)\delta$. By the previous lemma we know that if $d(x, W^u_{\delta^+}(0, f_t))$ is sufficiently small, then all $x_n, y_n, z_n$ are close to $W^u_{\delta^+}(0, f_t)$ such that

$$||y^u_n| - |z^s_n|| \leq d(y_n, z_n) < |z^s_n| - |z^s_{n-1}|,$$

and the amount $d(x, W^u_{\delta^+}(0, f_t))$ can be taken to be independent of $x$ and $t$. Then we get $|y^u_n| > |z^s_{n-1}|$. It implies $|y^u_i| > |z^s_{i-1}|$ for all $1 \leq i \leq n$ and in particular, $|y^u_1| > |z^u|$.

Since $y_1 \in S$, we have $|y^u_1| \leq \delta$ by the definition of $S$. Also, since $z, \bar{z} \in S^+$, we have $|z^u| \geq |\bar{z}^u| = \delta \geq |y^u_1|$. Therefore, by the fact $|y^u_i| > |z^s_{i-1}|$ and $|y^u_i| \leq |z^u|$, we can use the claim of the proof in the previous lemma with $\hat{y} = y_i$ and $\bar{z} = z_{i-1}$ to get inductively that $|y^u_i| \leq |z^u_{i-1}|$ for all $1 \leq i \leq n$. It follows that

$$\sum_{j=0}^i (|y^u_j| - |z^u_j|) = |y^u_0| - (|z^u_0| - |y^u_1|) - \cdots - (|z^u_{j-1}| - |y^u_j|) - |z^u_j| < |y^u_0| \leq \delta^+$$

for any $0 \leq j \leq n$.

Now we refine the estimates. Let

$$\tau = \{ x = (ty^u, x^s) \in W^u_{\delta^+}(x, f_t) : |z^u| \leq t|y^u| \leq |y^u| \}$$
and
\[ \tilde{\tau} = \{ \tilde{x} = (ty^u, \tilde{x}^s) \in W^u_\delta (x, f_t) : \delta \leq t|y^u| \leq \delta^+ \}. \]
Both \( \tau \) and \( \tilde{\tau} \) are curves in the unstable manifold \( W^u_\delta (x, f_t) \) since \( \{ ty^u \} \) forms a straight line in \( R^{n_u} \). Then we let \( \tilde{y} \) and \( \tilde{z} \) be the endpoints of the curve \( \tilde{\tau} \).

Let \( \pi^u : \Omega_1 \to R^{n_u} \) be the projection, and denote \( \tau^u = \pi^u \tau \) and \( \tilde{\tau}^u = \pi^u \tilde{\tau} \). We know that the first component of the the map \( f_t^j \) sends \( \tilde{\tau}^u \) to \( \tau^u \). Also recall that \( z^u \) and \( y^u \) are in the same direction. By (4.4) we know that all \( z_j^u \) and \( y_j^u \) are in the same direction as well. Hence \( \tau_j^u \) is a straight line. Let \( \ell \) denote the length. We get

\[
\|y_j^u| - |z_j^u\| = \ell(\tau_j^u) = \int_{\tau_j^u} \| Df_t^{-j} |_{R^{n_u}} \| dt(t).
\]

Also note that \( Df_t^j(x)|_{R^{n_u}} = Df_t(f_t^{-j-1}x)|_{R^{n_u}} \cdots Df_t(x)|_{R^{n_u}} \). By the fact that

\[
\| Df_t^{-j} |_{R^{n_u}} \|
\]

is uniformly bounded away from 0, and the \( C^2 \) norm of \( f_t \) is uniformly bounded from above on \( \Omega_0 \), we have that for \( j \leq n \),

\[
\log \frac{\| Df_t^{-j}(z) |_{R^{n_u}} \|}{\| Df_t^{-j}(y) |_{R^{n_u}} \|} \leq j \sum_{i=0}^{j-1} \left( 1 + \left( \frac{\| Df_t^{-1}(z_i) |_{R^{n_u}} \| - \| Df_t^{-1}(y_i) |_{R^{n_u}} \|}{\| Df_t^{-1}(y_i) |_{R^{n_u}} \|} \right) \right) \leq C \sum_{i=0}^{j-1} \frac{\| y_i^u \| - |z_i^u| \|}{\| Df_t^{-1}(y_i) |_{R^{n_u}} \|} \leq C D \sum_{i=0}^{j-1} \frac{\| y_i^u \| - |z_i^u| \|}{\| Df_t^{-1}(y_i) |_{R^{n_u}} \|} \leq C D \sum_{i=0}^{j-1} \frac{\| y_i^u \| - |z_i^u| \|}{\| Df_t^{-1}(y_i) |_{R^{n_u}} \|} \leq I_0.
\]

where we can take \( I_0 = \delta^+ C D \). Since this is true for any \( y, z \in \tilde{\tau} \), by (4.3) we obtain

\[
\frac{\| y_j^u \| - |z_j^u| \|}{\ell(\tau_j^u)} \leq e^{I_0} \frac{\| y_j^u \| - |z_j^u| \|}{\ell(\tilde{\tau}_j^u)}.
\]

Note that \( \ell(\tilde{\tau}^u) = \delta^+ - \delta \) is on the order of \( \| f_t(y^u) \| - |y^u| \), \( \ell(\tau^u) = d(y^u, z^u) \) is smaller than \( \delta^+ - \delta \), and \( \sum_{j=0}^{j-1} (\| y_i^u \| - |z_i^u| \|) \leq \delta^+ \), which is on order of \( |y| \).

By the construction of map \( f_t \), we have that \( |x^u|/(|f_t(x^u)| - |x^u|) \) is bounded by \( \left( |(f_t(x^u)| - |x^u|) \right)^{-r/(1+r)} \) up to a constant coefficient for all \( |x| \) close to 0. This is the case when \( \phi(x) \geq (1 + c|x|^r) + \) higher order terms. we get

\[
\sum_{j=0}^{j-1} \| y_j^u \| - |z_j^u| \| \leq e^{I_0} \delta^+ \frac{d(y^u, z^u)}{\delta^+ - \delta} \leq I_1 d(y^u, z^u) \frac{1}{1+r}
\]

for some \( I_1 \) independent of \( \delta \) and \( t \).

Now repeating the arguments as for (4.4), we get

\[
\log \frac{\| \det Df_t^{-j}(z) \|_{R^{n_u}}}{\| \det Df_t^{-j}(y) \|_{R^{n_u}}} \leq C D \sum_{i=0}^{j-1} d(y_i, z_i) \leq J' d(y^u, z^u) \frac{1}{1+r},
\]

where \( J' \) is independent of \( \delta \) and \( t \).

**Case two:** \( |y^u| = |z^u| \).
We may assume that $|y^s| \leq |z^s|$. Note that $\phi_t(r) = 1 + \psi'_t(r)$ and $\psi_t(r)$ is small as $t$ and $r$ are small. So by (4.2),
\[
|\det Df_t(x)|_{R^n} = 1 + \psi_t(|x^u|) + (d^u + s_t(x))|x^s|^2,
\]
where both $\psi_t(|x^u|)$ and $s_t(|x|)$ are small. So we know that there is $c_1 > 0$ of order $|z^s|$ such that if $|y^u| = |z^u|$, then
\[
|\det Df_t(z)^{-1} - \det Df(y)^{-1}|_{R^n} \leq c_1 (|z^s| - |y^s|).
\]
and therefore
\[
(4.5) \log \frac{|\det Df_t(z)^{-1}|_{R^n}}{|\det Df(y)^{-1}|_{R^n}} \leq c_1 C (|z^s| - |y^s|) \leq C_2 (|z^s| - |y^s|)^{1/(1+r)}
\]
Also, by (2.4), we know that there is a constant $c_0 > 0$ of order $|y^u||z^s|$ such that
\[
||\hat{z}_1^u| - |y_1^u|) \leq c_0 (|z^s| - |y^s|).
\]
Take $\hat{z}_1 = (\hat{z}_1^u, \hat{z}_1^s) \in W^n_u(x, f_t)$ such that $\hat{z}_1^u = (|y_1^u|/|z^u|)z^u$. That is, $\hat{z}_1^u$ is the point on $W^n_u(x, f_t)$ whose $u$-coordinate $\hat{z}_1^u$ is proportional to $z_1$ and satisfies $|\hat{z}_1^u| = |y_1^u|$. Hence, by (4.4) we have
\[
\log \frac{|\det Df_t^{-1}(z_1)|_{R^n}}{|\det Df_t^{-1}(z_1)|_{R^n}} \leq J'(|z_1^u| - |y_1^u|) \leq C_3 (|z^s| - |y^s|)^{1/(1+r)}.
\]
Therefore, by (4.5) and this inequality, we get
\[
\log \frac{|\det Df_t^{-n}(z)|_{R^n}}{|\det Df_t^{-n}(y)|_{R^n}} \leq \log \frac{|\det Df_t^{-n-1}(z_1)|_{R^n}}{|\det Df_t^{-n-1}(y_1)|_{R^n}} + (C_2 + C_3) (|z^s| - |y^s|)^{1/(1+r)}.
\]
So inductively,
\[
(4.6) \log \frac{|\det Df_t^{-n}(z)|_{R^n}}{|\det Df_t^{-n}(y)|_{R^n}} \leq (C_2 + C_3) \sum_{i=0}^{n-1} (|z_i^s| - |y_i^s|)^{1/(1+r)};
\]
where we regard $\hat{y}_0^s = y^s$, $\hat{y}_0^u = z^s$, and one of $\hat{z}_1^s$ and $\hat{y}_1^s$ are chosen in a similar way as we choose $\hat{z}_1^u$, depending which is larger between $\hat{z}_{i-1}^s$ and $\hat{y}_{i-1}^s$, and the other is the $s$-coordinate of the preimage of one of $\hat{z}_{i-1}^s$ and $\hat{y}_{i-1}^s$ that has smaller $s$-coordinate. Nevertheless, all $f_t^{-n-i}\hat{z}_i$ and $f_t^{-n-i}\hat{y}_i$ for all $0 \leq i \leq n-1$ are in $W^n_u(x_{n-1}, f_t)$. It is clear that
\[
d((f_t^{-n-i}\hat{y}_i)^u, (f_t^{-n-i}\hat{z}_i)^u) \leq d(y, z).
\]
So there is $C_4 > 0$ such that
\[
d((f_t^{-n+i}\hat{y}_i)^s, (f_t^{-n+i}\hat{z}_i)^s) \leq C_4 d(y, z) \quad \forall 0 \leq i \leq n-1
\]
because \( d((f^{-n+i}\tilde{y}_i)^*, (f^{-n+i}\tilde{z}_i)^*) \) is dominated by \( d((f^{-n+i}\tilde{y}_i)u, (f^{-n+i}\tilde{z}_i)u) \leq d(y, z) \) on unstable manifolds close to \( W^u_\delta(p, f_t) \). Since \( f_t \) is contracting in \( \mathbb{R}^n \) direction with the rate \( \kappa_n \), we get that

\[
d(\tilde{y}_i^*, \tilde{z}_i^*) \leq C_4(\kappa_n^i)^{n-i} d(y, z).
\]

Hence by (4.6), we get

\[
\log \frac{\left| \det Df^{-n}(z)|_{E^u_{f_t}} \right|}{\left| \det Df^{-n}(y)|_{E^u_{f_t}} \right|} \leq J d(y, z)^{1/(1+r)},
\]

(4.7)

The result of the lemma follows from (4.4) and (4.7), using the fact that the angles between \( E^u_{f_t}(f_t), E^u_{f_t}(f_t) \) and \( \mathbb{R}^n \) are exponentially decreasing as \( i \) change from \( n \) to 0. □

For any \( \delta > 0 \), let

\[
P_\delta(f_t) = [W^s_\delta(p, f_t), W^u_\delta(p, f_t)]
\]
a rectangle formed by stable and unstable manifolds, and

\[
P^+_\delta(f_t) = f_t P_\delta(f_t) \setminus P_\delta(f_t).
\]

It is easy to see that Lemma 4.2 holds if we replace \( S^+ \) by \( P^+_\delta(f_t) \).

**Proposition 4.3.** Given any \( \delta > 0 \), there exist constants \( \delta' > 0 \) and \( J > 1 \) such that for all \( f_t, t \in I \), and all \( x \in M^n \) with \( W^u_\delta(x, f_t) \cap P_\delta(f_t) = \emptyset \), for any \( y, z \in W^u_\delta(x, f_t) \) and \( n > 0 \),

\[
J^{-1} \leq \frac{\left| \det Df^{-n}(z)|_{E^u_{f_t}} \right|}{\left| \det Df^{-n}(y)|_{E^u_{f_t}} \right|} \leq J,
\]

(4.8)

Proof. Using Lemma 4.2 and the fact that \( f_t \) is uniformly hyperbolic outside \( P_\delta(f_t) \), we can get the result by the same arguments identical to that in the proof of Proposition 3.1 in [HY]. □

The inequalities (4.8) immediately lead to the following estimate of the density function of SRB measure on unstable manifolds of any fixed size that does not intersect the small rectangle \( P_\delta(f_t) \) at the origin.

**Corollary 1.** Let \( \mu_t \) be the SRB measure of \( f_t, t > 0 \). For any \( x \in \Lambda_{f_t} \), the hyperbolic attractor of \( f_t \), the density function \( \rho_{x,t} \) of the SRB measure \( \mu_t \) restricted to unstable manifolds satisfies

\[
J^{-1} \leq \frac{\rho_{x,t}(z)}{\rho_{x,t}(y)} \leq J,
\]

for \( y, z \in W^u(x, f_t) \) which does not intersect with \( P_\delta(f_t) \).
Proof. It is well known (see [LS]) that the density function \( \rho_{x,t} \) of the SRB measure \( \mu_t \) with respect to the Lebesgue measure \( m^u \) on the unstable manifold satisfies

\[
\frac{\rho_{x,t}(z)}{\rho_{x,t}(y)} = \prod_{i=0}^{\infty} \left| Df_t^{-1}(z_i) \right| E_{\mu_t}^u(f_t) \prod_{i=0}^{\infty} \left| Df_t^{-1}(y_i) \right| E_{\mu_t}^u(f_t)
\]

for all \( y, z \in W^u(x, f_t) \). Then we use Proposition 4.3. \( \square \)

5. Proof of Theorem A and its Corollary

A rectangle \( R \) is a set in \( M^n \) such that \( y, z \in R \) implies \([y, z], [z, y] \in R \). If \( \mathcal{D}^u, \mathcal{D}^s \) are pieces of \( W^u \)- and \( W^s \)-leaves for \( f_t \) respectively, then \([\mathcal{D}^u, \mathcal{D}^s] \) denotes the rectangle \( \{[y, z] : y \in \mathcal{D}^u, z \in \mathcal{D}^s \} \) provided that everything makes sense. If \( R \) is a rectangle and \( x \in R \), we let

\[
W^u(x, R, f_t) = W^u_\delta(x, f_t) \cap R \quad \text{and} \quad W^s(x, R, f_t) = W^s_\delta(x, f_t) \cap R.
\]

If \( Q \) and \( R \) are two rectangles, we say that \( f_t^n(Q) \) \( u \)-crosses \( R \) if \( \forall x \in Q \) with \( f_t^n x \in R \), \( f_t^n(W^u(x, Q)) \) \( \cap R = W^u(f_t^n x, R) \).

A Markov partition of \( f_t \) is a set in \( M^n \) \( \{R_i(f_t)\} \) on \( \int \) \( R_t(f_t) \) = \( R_t(f_t) \), such that (a) \( \int R_i(f_t) \cap \int R_j(f_t) = \emptyset \) whenever \( i \neq j \), and (b) \( f_t R_i(f_t) \) \( u \)-crosses \( R_j(f_t) \) and \( f_t^{-1} R_j(f_t) \) \( u \)-crosses \( R_i(f_t) \) whenever \( f_t R_i(f_t) \cap R_j(f_t) \neq \emptyset \).

We denote the rectangle containing the fixed point \( p \) by \( R_0(f_t) \) for \( 1 \leq t \leq 1 \). We may assume that \( R_0(f_t) \) contains a neighborhood of \( p \). Indeed, if we restrict the perturbation in a small neighborhood of \( p, R_0(f_t) \) is independent of \( 0 \leq t \leq 1 \).

Take \( \delta > 0 \) small enough such that \( B^u(p, \delta) \subset W^u(p, R_0(f_t)) \) for all \( 0 \leq t \leq 1 \). Let

\[
P_t = [B^u(p, \delta), W^s(p, R_0(f_t), f_t)],
\]

the rectangle determined by \( B^u(p, \delta) \) and \( W^s(p, R_0(f_t), f_t) \). Clearly, \( P_t \subset R_0(f_t) \) and \( W^s(x, R_0(f_t), f_t)) = W^s(x, P_t, f_t) \) if \( x \in P_t \) and \( W^s(x, R_0(f_t), f_t) \cap P_t = \emptyset \), otherwise.

Denote

\[
Q_t = f_{t}^{-1}P_t \setminus P_t.
\]

Since \( f_t \) has a hyperbolic attractor \( \Lambda_{f_t} \) on which \( f_t \) is topologically transitive, \( f_t \) has an SRB measure \( \mu_t \) on \( \Lambda_{f_t} \) for all \( 0 < t \leq 1 \). Let

\[
\nu_t = \frac{1}{\mu_t(\Lambda_t \setminus P_t)} \mu_t.
\]
We have \( \nu_t(\Lambda_t \setminus P_t) = 1 \).

**Lemma 5.1.** There is \( c > 0 \) such that \( \nu_t(Q_t) > c \) for all \( 0 < t \leq 1 \).

**Proof.** Suppose there exists \( \{ t_n \} \subset (0, 1] \) such that \( \nu_{t_n}(Q_{t_n}) \to 0 \).

Since each \( f_t \) is topologically transitive, for each rectangle \( R_i(f_t) \), \( i > 0 \), there is \( k = k(i) \geq 0 \) independent of \( t \) such that \( f_t^k R_i(f_t) \cap R_0(f_t) \neq \emptyset \). Note that if \( f_t^{k'} R_i(f_t) \cap R_0(f_t) \neq \emptyset \) for some \( k' > 0 \), then \( f_t^{k'} R_i(f_t) \) \( \mu \)-cross \( R_0(f_t) \) by the properties of Markov partition, and therefore \( f_t^{k'-1} R_i(f_t) \cap Q_t \neq \emptyset \). So we may assume that \( k(i) \) is chosen in such a way that \( f_t^k R_i(f_t) \cap R_0(f_t) = \emptyset \) for all \( j = 1, \cdots, k \).

Since the hyperbolicity of \( f_t \) is uniform outside \( R_0(f_t) \) for all \( 0 < t \leq 1 \), we know that there is \( c_1 = c_1(i) > 0 \) independent of \( t \) such that

\[
m^u(W^u(x, f_t^{-k} Q_t, f_t)) \geq c_1 m^u(W^u(x, R_i(f_t), f_t))
\]

for all \( x \in f_t^{-k} Q_t \cap R_i(f_t) \).

Denoted by \( \xi \) the partition of \( R_i(f_t) \) into the pieces of the unstable manifolds \( W^u(x, R_i(f_t), f_t) \) for \( f_t \), and denote by \( \nu^\xi_{x,t} \) be the corresponding conditional measure of \( \mu_t \) on \( \xi(x) \), where \( \xi(x) \) is the element of \( x \) containing \( x \). Note that \( \nu_t \) and \( \mu_t \) have the same conditional measure on \( \xi(x) \), \( \mu_t \)-a.e. \( x \in R_i(f_t) \).

Since \( \mu_t \) is an SRB measure, we can denote by \( \rho_{x,t} \) the density function of \( \nu^\xi_{x,t} \) with respect to the Lebesgue measure on \( W^u(x, R_i(f_t), f_t) \). We know that by Corollary 1 that the ratio \( \rho_{x,t}(y)/\rho_{x,t}(z) \) is bounded away from 0 and infinity for any \( y, z \in W^u(x, R_i(f_t), f_t) \), and the bounds can be chosen in a way that is independent of \( 0 < t \leq 1 \) and \( x \). So we know that there is \( c_2 > 0 \) such that

\[

\nu^\xi_{x,t}(W^u(x, f_t^{-k} Q_t, f_t)) \geq c_2 \nu^\xi_{x,t}(W^u(x, R_i(f_t), f_t)).

\]

Consequently, by the invariance of \( \nu_t \), we get

\[

\nu_t(Q_t) \geq \nu_t(f_t^{-k} Q_t \cap R_i(f_t)) \geq c_2 \nu_t(R_i(f_t)).
\]

That is,

\[

\nu_{t_n}(Q_{t_n}) \geq c_2 \nu_{t_n}(R_i(f_{t_n})).
\]

Note that \( P_t \subset R_0(f_t) \). Suppose that \( j = j_t \) is the smallest integer such that \( f_t^j P_t \) \( u \)-crosses \( R_0(f_t) \). Since \( f_t \) is uniformly hyperbolic \( P_t \), \( \{ j_t \} \) has a upper bound, we again denote it by \( j \). Suppose \( f_t R_0(f_t) \) is contained in \( R_{s_1}(f_t) \cup \cdots \cup R_{s_s}(f_t) \) for some \( s \). Then it is easy to see by the invariance of measure \( \nu_t \) that

\[

\nu_t(R_0(f_t) \setminus P_t) \leq j \sum \nu_t(R_{s_i}(f_t)).
\]


Since

\[ \nu_t(\cup_{i \neq 0} R_i(f_t)) + \nu_t(R_0(f_t) \setminus P_t) = 1 \]

for all \(0 < t \leq 1\), we know that \(\nu_t(\cup_{i \neq 0} R_i(f_t))\) is uniformly bounded away from 0 for all \(t\). Therefore, there is \(0 < c_3 < 1\) and \(i\) such that \(\nu_t(R_i(f_{tn})) \geq c_3\) for infinitely many \(n\). By taking a subsequence we may think that this is true for every \(n\). Hence we get

\[ \nu_t(Q_{tn}) \geq c_2c_3 > 0. \]

It is a contradiction. \(\Box\)

Put

\[ P_{0,t} = P_t \quad \text{and} \quad P_{i,t} = f_t^{-1}P_{i-1,t} \cap P_{i-1,t}, \quad \forall i = 1, 2, \cdots, \]

\[ Q_{i,t} = \{x \in Q_t : f^{j}x \in P_t \text{ for } j = 1, \cdots, i\}. \]

**Lemma 5.2.** There is \(C > 0\) such that for all \(t < (k_0 + i)^{-\beta}\),

\[ \frac{1}{C(k_0 + i)^{\beta n_u}} \leq \nu_t(Q_{i,t}) \leq \frac{C}{(k_0 + i)^{\beta n_u}}. \]

**Proof.** Note that for \(t < (k_0 + i)^{-\beta}\), \(W^u(p, P_{i,t}, f_t)\) is a ball of radius \((i + k_0)^{-\beta}\) on \(W^u(p, f_t)\). So we have \(m^u(W^u(p, P_{i,t}, f_t)) = C_1(i + k_0)^{-\beta n_u}\), where \(C_1\) is equal to the volume of the unit ball in \(n_u\) dimensional Euclidean space.

For any \(x \in Q_{i,t}\), consider the holonomy map \(\theta : W^u(x, Q_t, f_t) \to W^u(x, P_t, f_t)\) maps \(W^u(x, Q_{i,t}, f_t)\) to \(W^u(p, P_{i,t}, f_t)\). By Proposition 3.4, the stable foliation is absolute continuous with a uniform constant \(C_1 > 0\). So we get

\[ C_1^{-1} \leq m^u(W^u(x, Q_{i,t}, f_t)) \leq C_1. \]

Now we use the fact that the ratio \(\rho_t(y)/\rho_t(z)\) of the density of the conditional measure of \(\nu_{x,t}^E\) at any two points \(y, z \in W^u(x, Q_{i,t}, f_t)\) are uniformly bounded by constants \(C_2 > 0\), we get the result. \(\Box\)

**Proof of Theorem A.**

Since \(\mu_t\) is an absolutely continuous conditional measure on unstable manifolds \(W^u\) for \(f_t\), the entropy formula holds, that is,

\[ h_{\mu_t} = \int \lambda_t^+ d\mu_t, \]

where \(\lambda_t^+\) is the sum of all positive Lyapunov exponents. Further, it is well know that

\[ \int \lambda_t^+ d\mu_t = \int \log |\det Df_t|_{E^u(f_t)}|d\mu_t|. \]
So we only need show that

$$\int \log |\det Df|x_{E^0(f_t)}|d\mu_t \to 0 \quad \text{as} \quad t \to 0.$$ 

Take \(\varepsilon > 0\). Since \(Df_0(p)|_{E^0(f_0)} = \text{id}\), there is a neighborhood \(V\) of \(p\) on which \(\log |\det Df_t|_{E^0(f_t)}| \leq \varepsilon / 2\) for all small \(t > 0\).

It is easy to see that there is \(\delta > 0\) such that \(V\) contains the rectangle \([W^s_\delta(p, f_t), W^u_\delta(p, f_t)]\) for all \(0 < t \leq 1\), where \(W^s_\delta(p, f_t)\) denotes the local stable manifold of size \(\delta\) at \(p\) with respect to the map \(f_t\), and \(W^u_\delta(p, f_t)\) is understood in a similar way, though the latter is independent of \(t\). Take \(j > 0\) such that \(f_j^t(W^s(p, P_t)) \subset W^s_\delta(p, f_t)\) for all \(t \in I\). This is possible since \(f_t\) is uniformly contracting along stable direction. Also, take some \(k > 0\) such that \(\frac{1}{(k_0 + k)^\gamma} \leq \delta\), then \(W^u(p, P_{k,t}, f_t) \subset W^s_\delta(p, f_t)\) for all \(t \leq \delta\). Hence, we have \(f^jP_{k+j,t} \subset V\).

Note that \(Q_{1,t}\) is the set of points that enter \(P_t\) under the map \(f_t\), and \(P_t \setminus P_{1,t}\) is the set of points \(x\) that leave \(P_t\) under \(f_t\). So we have \(\nu Q_{1,t} = \nu_t(P_t \setminus P_{1,t})\). Similarly, we have \(\nu(Q_{i,t}) = \nu_t(P_{i-1,t} \setminus P_{i,t})\). Hence,

$$\nu_t(A_{f_t} \setminus P_{k+j,t}) = \nu_t(A_{f_t} \setminus P_t) + \sum_{i=1}^{k+j-1} \nu_t(Q_{i,t}).$$

By the definition of \(\nu_t\) we know that \(\nu_t(A_{f_t} \setminus P_{k+j,t}) = 1\) for all \(0 < t \leq 1\), and therefore by Lemma 5.2, \(\sum_{i=1}^{k+j-1} \nu_t(Q_{i,t})\) is uniformly bounded for all \(0 < t \leq 1\).

Also note that \(\beta n^u < 1\). So the series \(\sum_{i=1}^{\infty} (i + k_0)^{-\beta n^u}\) diverges. Therefore we can choose \(N > 0\) large enough such that if all \(\nu_t(Q_{i,t}), 1 \leq i \leq N\), satisfies the estimates in Lemma 5.2, then

$$\frac{\varepsilon}{2D} \sum_{i=k+j}^{N} \nu_t(Q_{i,t}) \geq \nu_t(A_{f_t} \setminus P_{k+j,t}),$$

where \(D = \max \{\log |\det Df_t(x)|_{E^0(f_t)} : x \in A_{f_t}\}\). Then we take

$$t_0 \leq (N + k_0)^{-\beta}.$$

So for any \(t \in (0, t_0)\), the above inequality holds, and therefore,

$$\frac{\varepsilon}{2D} \nu_{P_{k+j,t}} = \frac{\varepsilon}{2D} \sum_{i=k+j}^{\infty} \nu_t(Q_{i,t}) \geq \nu_t(A_{f_t} \setminus P_{k+j,t}).$$

Since

$$\mu_t = \mu_t(A_{f_t} \setminus P_t)^{-1} \nu_t,$$

we have

$$\frac{\varepsilon}{2D} \mu_t(P_{k+j,t}) \geq \nu_t(A_{f_t} \setminus P_{k+j,t}).$$
the above inequality also holds for \( \mu_t \) if \( t \in (0, t_0) \). Since
\[
\mu_t(P_{k+j,t}) + \mu_t(\Lambda_{jt} \setminus P_{k+j,t}) = 1,
\]
we get
\[
\mu_t(\Lambda_{jt} \setminus P_{k+j,t}) \leq \frac{\varepsilon}{2D}.
\]
It follows
\[
\mu_t(\Lambda_{jt} \setminus f_t^j P_{k+j,t}) \leq \frac{\varepsilon}{2D}.
\]
Recall \( f_t^j P_{k+j,t} \subset U \), in which \( \log |\det Df_t(x)|_{E^u(x)} \leq \varepsilon/2 \). We get that if \( t \in (0, t_0) \), then
\[
\int \log |\det Df_t|_{E^u(x)} d\mu_t = \int_{M^u \setminus f_t^j P_{k+j,t}} \log |\det Df_t|_{E^u(x)} d\mu_t + \int_{f_t^j P_{k+j,t}} \log |\det Df_t|_{E^u(x)} d\mu_t \leq D \cdot \mu_t(\Lambda_t \setminus f_t^j P_{k+j,t}) + \frac{\varepsilon}{2} \cdot \mu_t(f_t^j P_{k+j,t}) \leq D \cdot \frac{\varepsilon}{2D} + \frac{\varepsilon}{2} = \varepsilon.
\]
We proved the theorem. \( \square \)

**Corollary 2.** As \( t \to 0 \), \( \mu_t \to \delta_p \) in the week* topology, and
\[
h_{\mu_t}(f_t) \to 0 = h_{\delta_p}(f_0),
\]
where \( \delta_p \) is the Dirac measure at \( p \).

\( \square \)

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