RESEARCH PROBLEMS ASSOCIATED WITH GENERALIZATIONS OF $H^r$ FUNCTIONS AND BOUNDARY VALUES

Richard D. Carmichael
Department of Mathematics
Wake Forest University, Winston-Salem, NC 27109-7388, U.S.A.

ABSTRACT
Previously we have defined generalizations of $H^r$ functions in tubes and have obtained representation results for these functions for certain values of $r$ and for certain base sets of the tubes. For some subspaces of these generalized $H^r$ functions we have obtained the existence of ultradistributional boundary values for the elements in these subspaces. We also have defined Cauchy and Poisson integrals of ultradistributions and have obtained properties of these integrals. Here we survey previous results obtained concerning these topics and propose new research questions concerning each topic.

KEY WORDS: Analytic functions in tubes, boundary value, ultradistribution, Cauchy and Poisson integrals

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1 INTRODUCTION

Let $M_p$, $p = 0, 1, 2, 3, \ldots$, be a sequence of positive real numbers which satisfy some of the conditions (M.1), (M.2), (M.3), (M.2)', and (M.3)' as described in [7, p. 87]. For such a sequence $M_p$ put

$$M^*(\rho) = \sup_p \log(\rho^p!M_0/M_p), \ 0 < \rho < \infty,$$

which is the associated function of Komatsu [16].
Let \( B \) be a proper open subset of \( \mathbb{R}^n \) which does not contain the origin \( \bar{0} \in \mathbb{R}^n \). Let \( d(y) \) denote the distance from \( y \in B \) to the complement of \( B \) in \( \mathbb{R}^n \). We consider the norm growth

\[
\|f(x + iy)\|_{L^r} \leq K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \quad y \in B,
\]

where \( K > 0, T > 0, m \geq 0 \) and \( q \geq 0 \) are independent of \( y \in B \) and \( M^* \) is the associated function defined above. The analytic functions in the tube \( T^B = \mathbb{R}^n + iB \) which satisfy (1.1) will be designated as \( H^r_{M_p}(T^B) \), \( 0 < r < \infty \), functions where the sequence \( M_p \) satisfies (M.1) and (M.3)'.

The \( H^r_{M_p}(T^B) \) functions can be viewed as generalizations of the Hardy \( H^r \) functions. We have also studied other generalizations of the Hardy spaces in papers such as [1], [2], [3], [4], [6], and [14].

2 REPRESENTATION OF GENERALIZATIONS OF \( H^r \) FUNCTIONS

We have obtained Fourier-Laplace integral representations of the \( H^r_{M_p}(T^B) \) spaces in [5] and [9]. Unless stated otherwise \( B \) will denote a proper, open, connected subset of \( \mathbb{R}^n \) which does not contain \( \bar{0} \). The following two theorems are proved in [5].

**Theorem 2.1** Let \( f(z) \in H^r_{M_p}(T^B), 1 < r \leq 2 \). There exists a measurable function \( g(t), t \in \mathbb{R}^n, \) such that

\[
||e^{-2\pi <y,t>} g(t)||_{L^s} \leq K(1 + (d(y)))^{-m})^q \exp(M^*(T/|y|)), \quad y \in B,
\]

holds for \( 1/r + 1/s = 1 \); and

\[
f(z) = \int_{\mathbb{R}^n} g(t) e^{2\pi i <z,t>} dt, \quad z \in T^B.
\]

As a dual result to Theorem 2.1 we have the following.
Theorem 2.2 Let $1 < r \leq 2$ and let $g(t)$ be a measurable function satisfying (2.1) with $s$ replaced by $r$ there where $K > 0$, $T > 0$, $m \geq 0$ and $q \geq 0$ are independent of $y \in B$. We have

\[ f(z) = \int_{\mathbb{R}^n} g(t) e^{2\pi i <z,t>} dt \in H_{M^p}^s(T^B), \quad 1/r + 1/s = 1. \]

We desire to extend results like Theorems 2.1 and 2.2 to values of $r$ such that $2 < r < \infty$. For certain types of bases $B$ of the tube $T^B$ we have been able to do this for Theorem 2.1. Recall the $2^n$ n-rants $C_{u^n} = \{y \in \mathbb{R}^n : (-1)^u y_j > 0, j = 1, \ldots, n\}$ with $u = (u_1, u_2, \ldots, u_n)$ being any of the $2^n$ n-tuples whose components are either 0 or 1. For the types of cones called n-rant cones (quadrant cones) or polygonal cones we refer to [4]. An open convex cone in $\mathbb{R}^n$ which contains no entire straight line is called a regular cone. Recall the dual cone $C^*$ of a cone $C$; $C^* = \{t \in \mathbb{R}^n : <t, y> \geq 0, y \in C\}$.

For n-rants, n-rant cones (quadrant cones), polygonal cones, and regular cones we have results like Theorem 2.1 for $2 < r < \infty$. We first state such a result for the base $B$ in $T^B$ being any open convex cone that is contained in or is an n-rant.

Theorem 2.3 Let $C$ be an open convex cone which is contained in or is any of the $2^n$ n-rants $C_{u^n}$ in $\mathbb{R}^n$. Let $f(z) \in H_{M^p}^s(T^C)$, $2 < r < \infty$. There exists a measurable function $g(t)$, $t \in \mathbb{R}^n$, with support in $C^*$ almost everywhere such that

\[ ||e^{-2\pi< y,t>} g(t)||_{L^2} \leq K(1 + (d(y))^{-m})^q \exp(M^* T/|y|), \quad y \in C, \quad (2.3) \]

and

\[ f(z) = X(z) \int_{\mathbb{R}^n} g(t) e^{2\pi i <z,t>} dt, \quad z \in T^C, \quad (2.4) \]

where $X(z)$ is a polynomial in $z \in T^C$.

Theorem 2.3 can be extended to results of this type for $2 < r < \infty$ for n-rant cones (quadrant cones), polygonal cones, and regular cones. See [9, section 7] for details.
Several questions now arise from the results Theorems 2.1, 2.2, and 2.3. First, Theorem 2.3, the extension of Theorem 2.1 to $2 < r < \infty$, and the corresponding results for n-rant cones (quadrant cones), polygonal cones, and regular cones are not satisfactory in that we would like to have Theorem 2.3 proved for $f(z) \in H^r_{M_p}(T^B)$, $2 < r < \infty$, for the base $B$ being any proper, open, connected subset of $\mathbb{R}^n$ which does not contain $\bar{0} \in \mathbb{R}^n$. We ask if the representation (2.4) can be rewritten in the form $f(z) = \langle V, \exp(2\pi i \cdot < z, t >) \rangle$, $z \in T^B$, for some ultradistribution $V$? If so, $g(t)$ will have to possess sufficient properties to allow for $\langle V, \exp(2\pi i < z, t >) \rangle$ to be well defined.

We introduce the Cauchy and Poisson kernels corresponding to tubes $T^C$ in section 4 of this paper. Under certain circumstances we know that the functions $f(z) \in H^r_{M_p}(T^C)$ of Theorem 2.1, for $B = C$ being a regular cone, can be represented as a Cauchy integral; see [5, Corollary 2.2]. Is there a Cauchy integral representation or even a Poisson integral representation for the functions of Theorems 2.1 and 2.3 for $1 < r < \infty$ and tubes $T^C$ under the hypotheses stated in these theorems?

Theorem 2.2 is obtained for $1 < r \leq 2$. We would like to obtain this dual result to Theorem 2.1 for arbitrary $r$, $1 < r < \infty$, and to add Cauchy and Poisson integral representations for the $f(z)$ if possible.

We discuss boundary values of subspaces of the $H^r_{M_p}(T^B)$ spaces in the next section where we obtain more questions about these spaces for future research.

3 ULTRADISTRIBUTIONAL BOUNDARY VALUES

We refer the reader to [7, section 3] for the definitions of the ultradifferentiable functions $\mathcal{D}(*, L^s)$ and the ultradistributions $\mathcal{D}'(*, L^s)$, $1 \leq s \leq \infty$, where $*$ is either $(M_p)$ or $\{M_p\}$. Here also one can find properties of these spaces including characterization results for the ultradistribution spaces $\mathcal{D}'((M_p), L^s)$ and $\mathcal{D}'(\{M_p\}, L^s)$ of class $M_p$ of Beurling type and of Roumieu type, respectively.

Let the sequence of positive real numbers $M_p$, $p = 0, 1, 2, 3, \ldots$, satisfy (M.1), (M.2), and (M.3)’ and be such that $(M_p/p!)$ satisfies (M.1). The sequence $M_{p'} = (p!)^s$, $p = 0, 1, 2, 3, \ldots$, $s >$
1, is an example of a sequence which satisfies these hypotheses. The sequence $M_p$ will satisfy these hypotheses in this section.

Let $C$ be a regular cone in $\mathbb{R}^n$. We consider functions $f(z)$ which are analytic in $T^C = \mathbb{R}^n + iC$ and which satisfy

$$||f(x + iy)||_{L^r} \leq K \exp(M^*(T/|y|)), \ y \in C,$$  \hspace{1cm} (3.1)

where $K > 0$ and $T > 0$ are constants which are independent of $y \in C$ and $M^*$ is the associated function for the sequence $M_p$, $p = 0, 1, 2, \ldots$. Thus the norm growth that we are considering in (3.1) is a special case of that in (1.1) where $m = 0$ or $q = 0$.

In [8, section 5] and continuing in [13] we have proved the following ultradistributional boundary value result.

**Theorem 3.1** Let $f(z)$ be analytic in $T^C$ and satisfy (3.1) with $1 < r \leq 2$. There exists an element $U \in \mathcal{D}'((M_p), L^1)$ such that

$$\lim_{y \to 0} f(x + iy) = U$$  \hspace{1cm} (3.2)

in $\mathcal{D}'((M_p), L^1)$.

Recall the spaces $\mathcal{FD}(*, L^r)$ and $\mathcal{FD}'(*, L^r)$ which are defined in [8, section 2]. $\mathcal{FD}(*, L^r)$ is the Fourier transform space of $\mathcal{D}(*, L^r)$, $1 \leq r \leq 2$; and $\mathcal{FD}'(*, L^r)$ is the dual space of $\mathcal{FD}(*, L^r)$.

The following result, which is a dual theorem to Theorem 3.1, is proved in [8, section 5].

**Theorem 3.2** Let $1 < r \leq 2$. Let $g(t)$ be a measurable function on $\mathbb{R}^n$ such that

$$||e^{-2\pi<y,t>}g(t)||_{L^r} \leq K \exp(M^*(T/|y|)), \ y \in C,$$  \hspace{1cm} (3.3)
where $K > 0$ and $T > 0$ are constants which are independent of $y \in C$. We have

$$f(z) = \int_{\mathbb{R}^n} g(t) e^{2\pi i <z,t>} dt, \ z \in T^C, \ \ (3.4)$$

is analytic in $T^C$, satisfies (3.1) with $L'$ replaced by $L^s$, $1/r + 1/s = 1$; and there is an element $U \in \mathcal{D}'((M_p), L')$ such that

$$\lim_{y \to 0} f(x + iy) = U \ \ (3.5)$$

in $\mathcal{D}'((M_p), L')$. 

We have several desirable goals in extending the results of Theorems 3.1 and 3.2. Clearly we desire to extend results of this type for all values of $r$, $1 \leq r \leq \infty$. Further, we would like to obtain boundary value results of this type for arbitrary $H_{M_p}^r(T^C)$ functions in Theorem 3.1 and for the generalizing term $(1 + (d(y))^{-m})^q$ to be included in the bound on the right side of (3.3) in Theorem 3.2. We would like to prove these more general results to Theorems 3.1 and 3.2 for all $r$, $1 \leq r \leq \infty$. Additionally we desire to prove boundary value results like Theorems 3.1 and 3.2 and the extended proposed results of this paragraph for the space $\mathcal{D}'((M_p), L')$ as well.

For additional proposed boundary value research let us recall Theorem 2.3 and its extension to n-rant cones (quadrant cones), polygonal cones, and regular cones. For these types of cones we have obtained the representation (2.4) for $H_{M_p}^r(T^C)$, $2 < r < \infty$, functions. If $m = 0$ or $q = 0$ in the growth of the given $f(z) \in H_{M_p}^r(T^C)$, the function $g(t)$ in the representation (2.4) will satisfy (2.3) with $m = 0$ or $q = 0$ there. Thus the Fourier-Laplace integral in (2.4) obtains an ultradistributional boundary value as $y = \text{Im}(z) \to 0$, $y \in C$, by Theorem 3.2 since $g(t)$ satisfies (2.3) with $m = 0$ or $q = 0$ there and hence satisfies (3.3) with $r = 2$. Using this fact and the representation (2.4), can we obtain an ultradistributional boundary value result for the function $f(z)$ in Theorem 2.3 when $m = 0$ or $q = 0$ in the defining growth of $H_{M_p}^r(T^C)$? If we could do this, we could extend this boundary value property obtained for the case in Theorem
2.3 that $C$ is contained in or is any of the $2^n$ n-rants $C_n$ to obtain a similar boundary value property for each successive result like Theorem 2.3 for $C$ being an n-rant cone (quadrant cone), polygonal cone, and regular cone.

More generally we desire the boundary value questions posed in the preceding paragraph to be considered within the general setting of the problems posed in the paragraph before the preceding paragraph for general $H^r_{M_p}(T^n)$ functions, $1 < r < \infty$.

Carmichael and Pilipović [11] and Pilipović [17] have obtained boundary value results for $D'(*)_{L^r}$, $1 \leq r \leq \infty$, using quite different techniques than those used for Theorems 3.1 and 3.2. However, the analytic functions considered in [11] and [17] are taken to be defined in tubes in $\mathbb{C}^n$ corresponding to the n-rants $C_n$ and in half planes in $\mathbb{C}^1$ which are special cases of the general tube domain setting of Theorems 3.1 and 3.2. Hence the boundary results for $D'((M_p), L^r)$ obtained in Theorems 3.1 and 3.2 remain the most general such results in the setting of the analyticity of the functions considered. We desire to extend the boundary value results for $D'(*)_{L^r}$ of [11] and [17] as well as those of Theorems 3.1 and 3.2 to analytic functions in general tube domains for $1 \leq r \leq \infty$ and for as general a norm growth as possible such that the results of [11], [17], and Theorems 3.1 and 3.2 will be special cases.

We propose to study hyperfunctions associated with the analytic functions in this section and to obtain the singularity spectrum SSF or the analytic wave front set $WF_A(f)$ of the hyperfunction. See Hörmander [15] for these concepts.

4 CAUCHY AND POISSON KERNELS AND INTEGRALS

Let $C$ be a regular cone in $\mathbb{R}^n$, and let $C^*$ denote the dual cone of $C$ as defined previously in this paper. Put

$$K(z - t) = \int_{C^*} \exp(2\pi i < z - t, \eta>) d\eta, \quad z \in T^n, \quad t \in \mathbb{R}^n,$$

(4.1)

and

$$Q(z; t) = \frac{K(z - t) K(z - t)}{K(2iy)} = \frac{|K(z - t)|^2}{K(2iy)}, \quad z = x + iy \in T^n, \quad t \in \mathbb{R}^n.$$

(4.2)
\(K(z-t)\) and \(Q(z;t)\) are the Cauchy and Poisson kernel functions corresponding to the tube \(T^C\), respectively.

In [12] we have proved that \(K(z-t) \in D(\ast, L^s)\), \(1 < s \leq \infty\), as a function of \(t \in \mathbb{R}^n\) for \(z \in T^C\) and \(Q(z;t) \in D(\ast, L^s)\), \(1 \leq s \leq \infty\) as a function of \(t \in \mathbb{R}^n\) for \(z \in T^C\) where the sequence \(M_p\), \(p = 0, 1, 2, 3, \ldots\), in each case satisfies (M.1) and (M.3)’. We assume these two conditions on \(M_p\) throughout the remainder of this section.

Let \(U \in D(\ast, L^s)\), \(1 < s \leq \infty\). The Cauchy integral of \(U\) is

\[
C(U; z) = \langle U_t, K(z-t) \rangle, \quad z \in T^C. \tag{4.3}
\]

Let \(U \in D(\ast, L^s)\), \(1 \leq s \leq \infty\). The Poisson integral of \(U\) is

\[
P(U; z) = \langle U_t, Q(z;t) \rangle, \quad z \in T^C. \tag{4.4}
\]

In [7] we proved results concerning these Cauchy and Poisson integrals. We now extend some of these results to a more general setting and state problems that can be considered in future research.

Using the same proofs as in [7, Theorems 5.1 and 5.3], we now extend these results from the values \(2 \leq s < \infty\) to hold for all \(s\), \(1 < s < \infty\).

**Theorem 4.1** Let \(U \in D(\ast, L^s)\), \(1 < s < \infty\). The Cauchy integral \(C(U; z)\) is analytic in \(T^C\).

If \(U \in D((M_p), L^s)\), for each compact subcone \(C' \subset C\) there are constants \(R = R(n, C', s) > 0\) and \(T = T(C') > 0\) such that

\[
|C(U; z)| \leq R|y|^{-n/r} \exp(M^*(T/|y|)), \quad z = x + iy \in \mathbb{R}^n + iC', \tag{4.5}
\]

where \(n\) is the dimension, \(1/r + 1/s = 1\). If \(U \in D'(\{M_p\}, L^s)\), for each compact subcone \(C' \subset C\) and arbitrary constant \(T > 0\), which is independent of \(C' \subset C\), there is a constant \(R = R(n, C', s) > 0\) such that (4.5) holds.
Theorem 4.2  Let $U \in \mathcal{D}(\ast, L^s)$, $1 < s < \infty$. Let $\phi \in \mathcal{D}(\ast, L^{1})$. For fixed $y = \text{Im}(z) \in C$ we have

$$< C(U; x + iy), \phi(x) > = < U, < K(x + iy - t), \phi(x) >>. \quad (4.6)$$

We now desire to extend the results [7, Theorems 5.4, 5.5, 5.6, and 5.7] to hold for all $s$, $1 < s < \infty$, instead of just for $2 \leq s < \infty$. [7, Theorem 5.4] gives

$$< K(x + iy - t), \phi(x) > \to \mathcal{F}^{-1}[I_{C'}(\eta)\hat{\phi}(\eta); t], \phi \in \mathcal{D}((M_p), \mathbb{R}^n), \quad (4.7)$$
in $\mathcal{D}((M_p), L^s)$, $2 \leq s < \infty$, as $y \to 0$, $y \in C$. [7, Theorem 5.5] gives this same result holding in $\mathcal{D}((M_p), L^s)$, $2 \leq s < \infty$ for $\phi \in S_{\infty}(\{N_p\}, \{M_p\})$ or $\phi \in \mathcal{D}(\{M_p\}, \mathbb{R}^n)$. [7, Theorem 5.6] now yields

$$< C(U; x + iy), \phi(x) > \to < U, \mathcal{F}^{-1}[I_{C'}(\eta)\hat{\phi}(\eta); t] > \quad (4.8)$$
for $U \in \mathcal{D}'(\ast, L^s)$, $2 \leq s < \infty$, as $y \to 0$, $y \in C$, by combining the convergence (4.7) with (4.6) and the continuity of $U$. [7, Theorem 5.7] is an analytic decomposition theorem for $U \in \mathcal{D}'(\ast, L^s)$, $2 \leq s < \infty$, which is proved by combining the above stated analysis. We desire to extend all of these properties noted in this paragraph to all values of $s$, $1 < s < \infty$. For $1 < s < 2$, the techniques will probably need to be different from those used for $2 \leq s < \infty$ in the proofs of [7, Theorems 5.4 and 5.5] because of the extensive use of Fourier transform theory in the cases $2 \leq s < \infty$. We pose these desired extensions stated in this paragraph as problems for future research. (If [7, Theorems 5.4 and 5.5] can be extended to $1 < s < 2$ as desired, [7, Theorems 5.6 and 5.7] will then follow by the same proofs as before.)

In [8, Theorem 3.1] and [9, Theorem 4.1] we have given the proofs of each of the parts of the following theorem.

Theorem 4.3  If $U \in \mathcal{D}'((M_p), L^s)$, $2 \leq s < \infty$, for each compact subcone $C' \subset C$ there is a constant $T = T(C') > 0$ such that
\[
\|C(U; z)\|_{L^s} = \left( \int_{\mathbb{R}^n} |C(U; x + iy)|^s dx \right)^{1/s}
\] (4.9)

\[
\leq \begin{cases} 
K(U) \exp(M^*(T/|y|)), & y \in C' \subset C, \\
K(U, C', s, r, n)|y|^{-n(s-r)/rs \exp(M^*(T/|y|))}, & y \in C' \subset C, \text{ if } 2 < s < \infty,
\end{cases}
\]

1/r + 1/s = 1, where K(U) is a constant depending on U if s = 2 and K(U, C', s, r, n) is a constant depending on U, C', s, r, and n if 2 < s < \infty. If U ∈ \mathcal{D}'(\{M_p\}, L^s), 2 \leq s < \infty, for each compact subcone C' \subset C and arbitrary constant T > 0, which may or may not depend on C' \subset C, the inequality (4.9) holds where K(U) is a constant depending on U if s = 2 and K(U, C', s, r, n) is a constant depending on U, C', s, r, and n if 2 < s < \infty.

The proof of this result used Fourier transform theory extensively which may not be available in the same way for 1 < s < 2. We propose the problem to extend Theorem 4.3 to the cases 1 < s < 2.

We now turn to the Poisson integral given in (4.4). Since \(P(U; z)\) is now defined for \(U \in \mathcal{D}'(\ast, L^s), 1 \leq s \leq \infty\), the results [7, Theorems 6.1 and 6.3] hold now for 1 < s < 2 by the same proofs as before for the cases 2 \leq s < \infty. We now state these two extended results.

**Theorem 4.4** Let \(U \in \mathcal{D}'(\ast, L^s), 1 < s < \infty, \text{ and } \phi \in \mathcal{D}(\ast, L^1)\). We have

\[
< P(U; x + iy), \phi(x) > = < U_t, < Q(x + iy; t), \phi(x) >>, y \in C.
\] (4.10)

**Theorem 4.5** Let \(U \in \mathcal{D}'(\ast, L^s), 1 < s < \infty, \text{ and } \phi \in \mathcal{D}(\ast, \mathbb{R}^n)\). We have

\[
\lim_{y \to 0} \quad _{y \in C} < P(U; x + iy), \phi(x) > = < U, \phi >.
\]
5 GENERALIZATION OF A LEMMA

The result [10, Lemma 10] was important in order to obtain the main result of [10] which proved that a certain space of ultradistributions and a space of analytic functions were algebraically iso-
morphic. We desire to generalize [10, Lemma 10] for more general sequences \( M_p, p = 0, 1, 2, 3, \ldots \), and will use this generalization in future research of the type contained [10].

For the sequence \( M_p, p = 0, 1, 2, 3, \ldots \), of positive real numbers recall the associated function \( M^*(\rho) \) defined in section 1 of this paper. We state additional needed notation. From the sequence \( M_p \) define the sequences \( m_p = M_p/M_{p-1} \) and \( m_p^* = m_p/p, p = 1, 2, 3, \ldots \), as in [13, (2.2) and (2.3)]. We put \( m(\lambda) = (\)the number of \( m_p \leq \lambda \) as in [13, (2.4)], and we define the additional associated function \( M(\rho) \) as [13, (1.1)]

\[
M(\rho) = \sup_p \log(\rho^p M_0/M_p), 0 < \rho < \infty.
\]

Throughout the reminder of this section we assume that the sequence \( M_p \) satisfies (M.1),(M.2), (M.3)', \( M_p/p! \), \( p = 1, 2, 3, \ldots \), satisfies (M.1). We note that the condition (M.1) on \( M_p/p! \) is satisfied if and only if the sequence \( m_p^* \) defined in the preceding paragraph is nondecreasing.

Let \( \Gamma \) be a convex closed acute cone in \( \mathbb{R}^n \) with vertex at \( \bar{0} \in \mathbb{R}^n \). Let \( C \) be the interior of the dual cone \( \Gamma^* \) of \( \Gamma \). \( B(\bar{0}, a) \) will denote a closed ball with center at \( \bar{0} \in \mathbb{R}^n \) and radius \( a > 0 \). \( d(y) \) will denote the distance from \( y \in C \) to the boundary of \( C \) as noted before in this paper.

**Theorem 5.1** Let the sequence \( M_p \) satisfy (M.1), (M.2), (M.3)'; and let \( M_p/p! \) satisfy (M.1). Let \( a \geq 0, \epsilon > 0, \) and \( l > 0 \) be given real numbers. Let \( t \in \Gamma + B(\bar{0}, a) \). There exist constants \( K_1 > 0, K_2 > 0, l_1 > 0, \) and \( l_2 > 0 \) such that

\[
I(t) = \int_C \exp(-M^*(l/d(y)) - < y, t > - (a + \epsilon)|y|) \, dy \\
\geq K_1 \exp(-l_1 M(|t|)) \\
\geq K_2 \exp(-M(l_2 |t|)).
\]
Proof. Since $M_p/p!$ satisfying (M.1) is equivalent to the sequence $m_p^*$ being nondecreasing, the results [13, Lemmas 2.1, 2.2, 2.3, and 2.4] hold here. Let $C_1$ be a fixed compact subcone of $C$.

From [18, Chapter I, paragraph 4] there is a $\delta = \delta(C_1) > 0$ such that

$$d(y) = \inf_{\sigma \in \text{pr}(\Gamma)} \langle \sigma, y \rangle \geq \delta \inf_{\sigma \in \text{pr}(\Gamma)} |\sigma||y|, \quad y \in C_1,$$  \hspace{1cm} (5.2)

where $\text{pr}(\Gamma)$ denotes the projection of $\Gamma$ which is the intersection of $\Gamma$ with the unit sphere in $\mathbb{R}^n$. For the sequence element $m_1 = M_1/M_0$ and for $t \in \Gamma + B(\bar{0}, a)$ and $|t| > m_1 + 1$ put

$$A_1(t) = \left\{ y \in \mathbb{R}^n : |y| \in \left[ \frac{(2sl)/\delta M(|t|)}{|t|}, \frac{(2sl)/\delta M(|t|) + 1}{|t|} \right] \right\},$$

where $\delta$ is from (5.2) and $s = 2(m_1 + 1)$. Now put

$$C_1(t) = A_1(t) \cap C_1.$$

From (5.2) we have

$$\frac{l}{d(y)} \leq \frac{l}{|y|} \leq \frac{|t|}{2sM(|t|)}, \quad y \in C_1(t), \quad t \in \Gamma + B(\bar{0}, a), \quad |t| > m_1 + 1.$$ \hspace{1cm} (5.3)

Since $\langle y, t \rangle = |y||t| \cos(\theta_{y,t})$ then $\langle y, t \rangle \leq |y||t|$ and $-\langle y, t \rangle \geq -|y||t|$. For $|t| > m_1 + 1$ and $t \in \Gamma + B(\bar{0}, a)$ we use (5.3), the definition of $A_1(t)$, and the fact that $M^*(\rho)$ is a nondecreasing function to obtain

$$I(t) > \int_{C_1(t)} \exp(-M^*(l/d(y))) - |y||t| - (a + \epsilon)|y| dy$$

$$= \int_{C_1(t)} \exp(-M^*(l/d(y))) - (a + \epsilon + |t|)|y| dy$$

$$\geq \exp\left(-M^*\left(\frac{|t|}{2sM(|t|)}\right)\right) \exp\left(-(a + \epsilon + |t|)\left(\frac{(2sl)/\delta M(|t|) + 1}{|t|}\right)\right) \int_{C_1(t)} 1 dy.$$
By [13, Lemma 2.4], where $s = 2(m_1 + 1)$, we have

$$\exp \left( -M^* \left( \frac{|t|}{2sM(|t|)} \right) \right) \geq \exp(-M(|t|)) \exp(A), \ |t| > m_1 + 1, \quad (5.5)$$

for some constant $A$ depending on $m_1$. Now we obtain a constant $Q > 0$ depending on $n$ and $C_1$ such that

$$\int_{C_1(t)} 1 \, dy = \int_{\text{pr}(C_1)} \int_{[(2s)/|t|]M(|t|) + 1}^{[(2s)/|t|]M(|t|) + 1} u^{n-1} \, du \, d\sigma$$

$$= \left( \int_{\text{pr}(C_1)} 1 \, d\sigma \right) (1/n) \left( \left( \frac{(2s)/|t|}{M(|t|)} \right) + 1 \right)^n - \left( \frac{(2s)/|t|}{M(|t|)} \right)^n \right)$$

$$> (1/n) \left( \int_{\text{pr}(C_1)} 1 \, d\sigma \right) |t|^{-n} \quad (5.6)$$

$$\geq Q \exp(-M(|t|))$$

for $|t| > m_1 + 1$ and $t \in \Gamma + B(\bar{0}, a)$. Since $M_p$ satisfies (M.1) and (M.3)' here we have from [16, Lemma 4.1, pp. 55-56] that $M(|t|)/|t| \to 0$ as $|t| \to \infty$. Using this fact we have the existence of a constant $Q'$ depending on $\delta$ and $C_1$ such that

$$\exp \left( -(a + \epsilon) 2sM(|t|)/\delta|t| \right) \geq Q', \ |t| > m_1 + 1. \quad (5.7)$$

Using (5.5), (5.6), and (5.7) in (5.4) we have for $|t| > m_1 + 1$ and $t \in \Gamma + B(\bar{0}, a)$ that

$$I(t) \geq \exp(A) \exp(-M(|t|)) Q \exp(-M(|t|)) Q' \exp(-2sM(|t|)/\delta) \exp(-(a + \epsilon)/(m_1 + 1)) e^{-1}$$

$$= B \exp(-(2 + (2s)/\delta)M(|t|)) \quad (5.8)$$

where $B$ depends on $m_1$, $n$, and $C_1$ but not on $t$.

Now let $t \in \Gamma + B(\bar{0}, a)$ with $|t| \leq m_1 + 1$ and put

$$C_2(t) = A_2(t) \cap C_1$$

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where

\[ A_2(t) = \left\{ y \in \mathbb{R}^n : |y| \in \left[ \frac{(2sl)/\delta M(|t| + m_1 + 1)}{|t| + m_1 + 1}, \frac{(2sl)/\delta M(|t| + m_1 + 1) + 1}{|t| + m_1 + 1} \right] \right\}. \]

From (5.2) we obtain

\[ \frac{l}{d(y)} \leq \frac{l\delta(|t| + m_1 + 1)}{\delta 2sM(|t| + m_1 + 1)} = \frac{|t| + m_1 + 1}{2sM(|t| + m_1 + 1)}, y \in C_2(t). \quad (5.9) \]

For \(|t| \leq m_1 + 1\) we have

\[ \exp \left( -M^* \left( \frac{|t| + m_1 + 1}{2sM(|t| + m_1 + 1)} \right) \right) \geq \exp \left( -M^* \left( \frac{2m_1 + 2}{2sM(m_1 + 1)} \right) \right). \quad (5.10) \]

For \(y \in C_2(t)\) with \(|t| \leq m_1 + 1\) we have

\[ \exp \left( -(a + \epsilon + |t|)|y| \right) \geq \exp \left( -(a + \epsilon + m_1 + 1) \left( \frac{(2sl)/\delta M(2m_1 + 2) + 1}{m_1 + 1} \right) \right). \quad (5.11) \]

Further

\[ \int_{C_2(t)} 1 \, dy = \int_{pr(C_2)} \frac{(2sl)/\delta M(|t| + m_1 + 1) + 1}{|t| + m_1 + 1} \frac{u^{n-1}}{1/n} du d\sigma \]

\[ = \left( \int_{pr(C_2)} 1 \, d\sigma \right) \left( \frac{1/n}{(2sl)/\delta M(|t| + m_1 + 1) + 1} \right) - \left( \frac{(2sl)/\delta M(|t| + m_1 + 1) + 1}{|t| + m_1 + 1} \right)^n \]

\[ \geq \left( \int_{pr(C_2)} 1 \, d\sigma \right) \left( \frac{1/n}{2m_1 + 2} \right) \left( |t| + m_1 + 1 \right)^{-n} \quad (5.12) \]

Using (5.9), (5.10), (5.11), and (5.12) we thus have for \(t \in \Gamma + B(\bar{0}, a)\) with \(|t| \leq m_1 + 1\) that
\[ I(t) \geq \int_{C_2(t)} \exp(-M^*(l/d(y)) - (a + \epsilon + |t||y|)dy \\
\geq \exp\left(-M^*\left(\frac{2m_1 + 2}{2sM(m_1 + 1)}\right)\exp\left(-(a + \epsilon + m_1 + 1)\left(\frac{(2sl)/\delta M(2m_1 + 2) + 1}{m_1 + 1}\right)\right)\right) \\
\quad \cdot \left(\int_{pr(C_2)} 1 d\sigma\right) (1/n) (2m_1 + 2)^{-n} \\
= R \] (5.13)

where \( R \) depends on \( m_1, C_1, \) and \( n \) in addition to the given \( a \geq 0, l > 0, \) and \( \epsilon > 0. \)

Combining (5.8) and (5.13) we obtain the first inequality in (5.1) with \( l_1 = 2 + (2sl)/\delta > 0. \)

Since the sequence \( M_p \) satisfies (M.1) and (M.2) here by hypothesis, \([8, (2.9)]\) holds for \( L = l_1; \) and the second inequality in (5.1) follows immediately from the first inequality for some \( K_2 > 0 \) and \( l_2 > 0. \)
References


