BOUNDS FOR INVERSES OF TRIANGULAR TOEPLITZ MATRICES
KENNETH S. BERENHAUT, DANIEL C. MORTON AND PRESTON T. FLETCHER *

Abstract. This short note provides an improvement on a recent result of Vecchio on a norm bound for the inverse of a lower triangular Toeplitz matrix with nonnegative entries. A sharper asymptotic bound is obtained as well as a version for matrices of finite order. The results are shown to be nearly best possible under the given constraints.

Key words. Toeplitz matrix, inverse matrix, recurrence relation.

AMS subject classifications. 15A09, 39A10, 15A57, 15A60

1. Introduction. This paper provides an improvement on a recent result of Vecchio on inverses of lower triangular Toeplitz matrices. We refer the reader to Vecchio [13] for discussion of applications particularly those to stability analysis of linear methods for solving Volterra integral equations. Example 1, below, displays an improvement in that realm. Other references on the topic, mentioned in [13], include [1]–[2], [7], and [10]–[12].

The matrices of interest here are \((n+1)\times (n+1)\) truncations of infinite lower triangular matrices generated by sequences \(\{a_i\}_{i\geq 0}\), ie.

\[
A_n = \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_n \\
a_1 & a_0 & a_1 & \cdots & a_{n-1} \\
a_2 & a_1 & a_0 & \cdots & a_{n-2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_n & \cdots & a_1 & a_0 & a_0
\end{bmatrix},
\]

(1.1)

We will prove the following theorem.

THEOREM 1.1. Suppose that the sequence \(\{a_i\}_{i\geq 0}\) satisfies

\[
a_0 \geq a_1 \geq a_2 \geq \cdots a_n \geq a \geq 0,
\]

(1.2)

for some constant \(a\) and all \(n\). Then

\[
\|A_n^{-1}\|_\infty \leq \begin{cases} 
\frac{a}{n} \left( 1 - \left( 1 - \frac{a}{a_0} \right)^{\left[ \frac{n}{2} \right]+1} \right), & \text{if } a > 0 \\
\frac{2a}{n} \left( \left[ \frac{n}{2} \right] + 1 \right), & \text{if } a = 0
\end{cases}
\]

(1.3)

and, in particular if \(a > 0\)

\[
\|A_n^{-1}\|_\infty \leq \frac{2}{a},
\]

(1.4)

independent of \(a_0\).

Note that Vecchio [13] obtained the comparable (to (1.4)) though less sharp result

\[
\|A_n^{-1}\|_\infty \leq \frac{2}{a} + \frac{1}{a_0},
\]

(1.5)

*Wake Forest University, Department of Mathematics, Winston-Salem, NC 27109 (berenhks@wfu.edu, mortdc@wfu.edu and fletpt1@wfu.edu). Revised: February 27, 2005.
and the methods used therein do not seem amenable to a bound for finite \(n\) as in (1.3). Since we are primarily interested in reliable and applicable, explicit bounds for finite \(n\), standard Lyapunov stability methods (cf. [3]–[6], [8]–[9], and the references therein) are also not directly applicable.

Note: (Optimality of Theorem 1.1). Suppose \(a_1 = a_0 > 0\) and \(a_i \equiv a\) for \(2 \leq i \leq n\). Also, for \(0 \leq i \leq n\) let

\[
b_i = (-1)^i \frac{1}{a_0} \left( 1 - \frac{a}{a_0} \right)^{\left\lfloor \frac{i}{2} \right\rfloor}.
\]

It is easy to verify that in this case,

\[
A_n^{-1} = \begin{bmatrix}
b_0 & & \\
b_1 & b_0 & \\
b_2 & b_1 & b_0 \\
\vdots & \ddots & \ddots & \ddots \\
b_n & \cdots & b_1 & b_0
\end{bmatrix},
\]

and hence if \(a > 0\),

\[
\|A_n^{-1}\|_{\infty} = \sum_{i=0}^{n} |b_i| = \sum_{i=0}^{n} \frac{1}{a_0} \left( 1 - \frac{a}{a_0} \right)^{\left\lfloor \frac{i}{2} \right\rfloor} = \frac{2}{a} \left( 1 - \left( 1 - \frac{a}{a_0} \right)^{\left\lfloor \frac{n}{2} \right\rfloor+1} \right) - \begin{cases} 0 & \text{if } n \text{ is odd} \\ |b_n| & \text{if } n \text{ is even} \end{cases}.
\]

If \(a = 0\), \(b_i = \frac{1}{a_0} (-1)^i\) and hence

\[
\|A_n^{-1}\|_{\infty} = \sum_{i=0}^{n} |b_i| = \frac{n + 1}{a_0} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{a_0} & \text{if } n \text{ is even} \end{cases}.
\]

Note that (1.8) and (1.9) compare quite favorably with (1.3), and that for \(a > 0\), (1.4) is in fact optimal under the monotonicity assumption in (1.2).

We close this section with an example.

Example 1. In [13], the difference, \(E_n\), between the analytical and numerical solutions due to the application of a Direct Quadrature (DQ) method to the Volterra integral equation (VIE)

\[
y(t) = g(t) + \int_{0}^{t} k(t - s)y(s)ds, \quad t \in [0, T], y, g, k \in \mathbb{R}
\]

was considered.
In particular, it was shown that if $k(t) \leq 0 \leq k'(t)$, for $t \geq 0$, $\lim_{t \to \infty} k(t) = \tilde{k} < 0$ and $|k(0)| < c_1$, then

\begin{equation}
\|E_n\|_\infty \leq \left( -\frac{2}{\tilde{k}} + \frac{1}{c_2} \right) \|\Gamma_n\|_\infty
\end{equation}

where $E_n = A_n\Gamma_n$, and $c_1$, $c_2$ and $A_n$ vary according to the DQ method used (see [13]).

Applying (1.3) in place of (1.4) gives

\begin{equation}
\|E_n\|_\infty \leq -\frac{2}{\tilde{k}} \left( 1 - \left( 1 + \frac{\tilde{k}}{c_2} \right)[\frac{1}{2}]^{i+1} \right) \|\Gamma_n\|_\infty
\end{equation}

\begin{equation}
\leq -\frac{2}{\tilde{k}} \|\Gamma_n\|_\infty.
\end{equation}

The next section comprises a proof of Theorem 1.1.

2. Proof of Theorem 1.1. As in Vecchio [13] we have that the entries in $A_n$ and $A_n^{-1}$ are related by a linear recurrence with $b_0 = \frac{a_0}{a_0}$ and

\begin{equation}
b_i = \sum_{j=0}^{i-1} -\gamma_{i-1-j} b_j
\end{equation}

for $1 \leq i \leq n$, where

\begin{equation}\gamma_i = \frac{a_{i+1}}{a_0}.
\end{equation}

Note that if we set

\begin{equation}
h_i = a_0 b_i
\end{equation}

for $0 \leq i \leq n$ then $h_0 = 1$ and

\begin{equation}
h_i = \sum_{j=0}^{i-1} -\gamma_{i-1-j} h_j
\end{equation}

for $1 \leq i \leq n$, and hence we will restrict attention to the simpler sequence $\{h_i\}$.

Also, define the nonnegative sequence $\{S_i\}$ by $S_i = 0$ for $i \leq -2$, $S_{-1} = 1$ and

\begin{equation}
S_i = \sum_{j=-1}^{i-1} d_{i-1-j} S_j,
\end{equation}

for $i \geq 0$. 


A key to obtaining our bound will be a splitting of $h_i$ into positive and negative parts as given by Lemma 2.1, below. Note that Equation (2.8) may be compared with the use of the fundamental matrix $\{u_j\}$ in [13], given by

$$u_n = 1 - \sum_{l=0}^{n-1} \gamma_{n-1-l} u_l,$$

for $n \geq 0$. In fact, it is not difficult to show that $S_i = u_{i+1}$ for all $i \geq -1$. The following lemma follows from (2.6) in [13]. For completeness, we give a proof here.

**Lemma 2.1.** We have

$$h_i = S_{i-1} - S_{i-2},$$

for $0 \leq i \leq n$.

**Proof.** Note that taking differences in (2.6) gives,

$$S_i - S_{i-1} = \sum_{j=0}^{i} d_{i-j} (S_{j-1} - S_{j-2}),$$

for $i \geq 1$. As well, by (2.4),

$$h_{i+1} - h_i = \sum_{j=0}^{i} (\gamma_{i-1-j} - \gamma_{i-j}) h_j - \gamma_0 h_i,$$

for $i \geq 1$, and hence

$$h_{i+1} = \sum_{j=0}^{i} (\gamma_{i-1-j} - \gamma_{i-j}) h_j + (1 - \gamma_0) h_i$$

$$= \sum_{j=0}^{i} d_{i-j} h_j.$$

Since $S_{-1} - S_{-2} = 1 = h_0$ and $S_0 - S_{-1} = d_0 - 1 = h_1$, $\{S_j - S_{j-1}\}_{j \geq -1}$ and $\{h_j\}_{j \geq 0}$ satisfy the same recurrence and the lemma follows.

Now, let $c_m = \sum_{j=0}^{m} d_j = 1 - \gamma_m = 1 - a_{m+1}/a_0$. We require the following lemma concerning alternating sums over subsequences of $\{S_i\}$.

**Lemma 2.2.** If $-2 \leq i_1 < i_2 < \cdots < i_k = m$ is some increasing, finite, integer sequence then

$$\sum_{j=1}^{k} (-1)^i S_{i_j} \leq \sum_{j=0}^{[m+1]} c_j^j.$$

**Proof.** If $m \leq 0$, (2.12) is immediate. Suppose (2.12) holds for $m < N$ where $N > 0$. For $m = N$, we have

$$\sum_{j=1}^{k} (-1)^i S_{i_j} = \sum_{j=1}^{k} (-1)^i \sum_{v=0}^{N} d_v S_{i_j-v-1}$$
= \sum_{v=0}^{N} d_v \sum_{j=1}^{k} (-1)^j S_{j-v-1}
\leq \sum_{v=0}^{N} d_v \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} c_j^v
\leq \sum_{v=0}^{N} d_v \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} c_j^v
\leq e_m \sum_{j=0}^{\lfloor \frac{N+1}{2} \rfloor} c_j^m
\leq \sum_{j=0}^{\lfloor \frac{N+1}{2} \rfloor} c_j^m.
(2.13)

where the first inequality in (2.13) follows by induction.  

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** From Lemma 2.1, we have

\[
\sum_{j=0}^{n} |h_j| = \sum_{j=0}^{n} |S_{j-1} - S_{j-2}|
= 1 + \sum_{j=0}^{n-1} (-1)^j (S_j - S_{j-1})
= 2S_{-1} + \sum_{j=0}^{n-2} Z_j S_j + (-1)^{n-1} S_{n-1},
\]
(2.14)

where for 0 \leq j \leq n - 1,

\[
l_j = \begin{cases} 0, & \text{if } S_j \geq S_{j-1} \\ 1, & \text{if } S_j < S_{j-1} \end{cases}
\]
(2.15)

and for 0 \leq j \leq n - 2,

\[
Z_j = (-1)^{l_j} - (-1)^{l_{j+1}} \begin{cases} 2, & \text{if } l_j = 0 \text{ and } l_{j+1} = 1 \\ 0, & \text{if } l_j = l_{j+1} \\ -2, & \text{if } l_j = 1 \text{ and } l_{j+1} = 0 \end{cases}.
\]
(2.16)

Equation (2.14) and Lemma 2.2 give for some sequence -2 \leq i_1 < i_2 < \cdots < i_k = n - 1,

\[
\sum_{j=0}^{n} |h_j| = 2 \left( S_{-1} + \sum_{j=1}^{k-1} (-1)^{i_j} S_{i_j} \right) + (-1)^{n-1} S_{n-1}
\leq 2 \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} c_{n-1}^j.
\]
(2.17)
Hence from (2.3) and (2.17),

\[
\|A_n^{-1}\|_\infty = \sum_{j=0}^{n} |b_j| \\
= \sum_{j=0}^{n} \frac{1}{a_0} |h_j| \\
\leq \frac{2}{a_0} \sum_{j=0}^{[\frac{n}{2}]} c_{n-1} \\
= \frac{2}{a_0} \sum_{j=0}^{[\frac{n}{2}]} \left( 1 - \frac{a_n}{a_0} \right)^j \\
\leq \frac{2}{a_0} \sum_{j=0}^{[\frac{n}{2}]} \left( 1 - \frac{a_n}{a_0} \right)^j.
\]

(2.18)

(2.19)

and (1.3) and (1.4) follow.

**Remark.** Note that if knowledge of \(a_n\) is available, then (2.18) provides an improvement on (2.19).

**Acknowledgments.** The first author acknowledges financial support from a Sterge Faculty Fellowship.

We are very thankful to a referee for comments and insights that improved this manuscript.

**REFERENCES**


