Parsimonious and Completely Compatible Taper, Total, and Merchantable Volume Models

Lewis Jordan, Kenneth Berenhaut, Ray Souter, Richard F. Daniels

Abstract: A parsimonious and compatible system of taper, total, and merchantable volume equations sharing a common vector of three parameter estimates was derived from Clark and Saucier’s (1990) volume ratio model. Clark and Saucier’s model expresses volume inside bark as a function of outside bark diameter and is itself not compatible with the derived taper equation. To ensure compatibility, models were fit to express volume outside bark as functions of diameter outside bark. By constraining the limits of integration of the taper function to go from the tip to groundline diameter, a completely compatible system of taper, total, and merchantable volume equations were developed. Systems of equations were fit using nonlinear seemingly unrelated regressions with nonlinear cross-equation constraints to account for contemporaneous correlations in the data. The system fit was found to account for at least 96% of the variation in diameter outside bark, total, and merchantable outside bark volume. For. Sci. 51(6):578–584.

Key Words: Compatible equations, merchantable volume, seemingly unrelated regression, taper.

The development of stem profile taper and merchantable volume models are common in forestry related literature. Numerous taper and subsequently derived merchantable volume functions of various forms have been developed especially in the last 30 years, from simple taper models (Kozak et al. 1969, Ormerod 1973, Hilt 1980), segmented polynomial models (Bruce et al. 1968, Max and Burkhart 1976, Cao et al. 1980), and geometry-oriented models (Parresol and Thomas 1996, Fang and Bailey 1999, Zhang et al. 2002). Newnham (1988) states that the two reasons for continuous study in this area are: 1) because no single theory has been developed that adequately explains the variation in stem form for all kinds of trees, and 2) as a method of estimating volume, a single taper equation can estimate both total and merchantable tree stem volume. If tree form can be accurately described, then volume for any merchantability limit can be accurately predicted.

Merchantable volume equations allow for the prediction of volume to a specified upper stem diameter. For utilization purposes, it is desirable to merchandise trees into multiple products, hence the development of a taper function is essential (McTague and Bailey 1987). Clutter (1980) showed that, for a given merchantable volume equation, there is an intrinsically defined compatible taper function. This idea implies that integration of the taper function from the ground to total height of the tree would return the appropriate volume, and subsequently the merchantable volume equation. Clutter (1980) states “The accuracy and precision of such taper functions will depend on the accuracy and precision of the merchantable volume equations from which they are derived.” Conversely, the same holds true for merchantable volume equations derived from taper functions. Precision and accuracy are a direct result of the model forms, be they empirical or geometry-oriented taper and volume functions, that best fit a unique data set.

Burkhart (1977) developed a merchantable volume ratio equation based on upper stem diameter, which eliminated the need for separate volume equations for differing stem diameters. Cao and Burkhart (1980) developed a volume ratio model based on the distance from the tree top to a merchantable height. Knoebel et al. (1984) developed a technique to derive compatible taper and merchantable height functions from the Burkhart (1977) and Cao and Burkhart (1980) volume ratio models. However, McTague and Bailey (1987) showed that the integration of the Knoebel et al. (1984) taper function from the groundline to some merchantable height does not equal the value of merchantable volume from the Cao and Burkhart (1980) ratio model.

Van Deusen et al. (1981), Clark and Saucier (1990), and Tasissa et al. (1997) presented exponential volume ratio models for the prediction of stem volume to some

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merchandable diameter limit. The exponential ratio models possess desirable properties such that, as diameter outside bark approaches infinity, the ratio goes to zero, thus predicted that volume goes to zero. Secondly, when diameter outside bark is equal to zero, the ratio becomes one, thus volume is equal to total stem volume. Clark and Saucier (1990) introduced a widely used and accepted merchantable tree volume exponential ratio equation for plantation-grown loblolly pine (Pinus taeda L.) in the southeastern United States. Clark and Saucier’s (1990) exponential ratio model was developed to predict merchantable volume inside bark as a function of diameter outside bark. In itself, Clark and Saucier’s (1990) model can not be differentiated to a compatible taper equation, because volume inside bark is calculated as a function of outside bark diameter.

The purpose of this article is derivation of a compatible system of taper and merchantable volume models from the popular exponential ratio model presented by Clark and Saucier (1990). To ensure compatibility, models were fit to express volume outside bark as a function of diameter outside bark.

**Derivation**

The general form of the model is given as

\[ V_{\text{DB}} = V_T V_R, \]  
(1)

where, \( V_{\text{DB}} \) = merchantable inside bark volume (m³) to an outside bark top diameter (D_m) (cm); \( V_T = f(T) \) = total inside bark stem volume, where \( D \) is dbh (cm), and \( H \) is total height (m); \( V_R = \exp(\beta_1 D_m^\beta_2 D^\beta_3) \) = the volume ratio, where \( e \) is the base of the natural logarithm, and \( \phi = (\beta_1, \beta_2, \beta_3)^T \) and \( \beta = (\beta_1, \beta_2, \beta_3)^T \) are coefficients to be estimated.

The objective is to express outside bark diameter (d_o) (cm) as a function of distance from the top of the stem \( t \). Thus, the taper function to be derived is given as

\[ d_o^2 = f(t). \]  
(2)

Let, \( T = \) the distance from the top of the stem to the merchantability limit \( D_m \), and \( M = \) the merchantable height (distance from the ground to the height of the merchantability limit \( D_m \)), so that \( H = T + M \). By definition of the taper function,

\[ D_m^2 = f(T). \]  
(3)

It follows from Clutter (1980) that,

\[ k \int_0^T f(t) \, dt = V_T - V_T \exp(\beta_1 D_m^\beta_2 D^\beta_3), \]  
(4)

where \( k = \pi/40000 \), for conventional metric measurement units. Substituting for \( D_m^2 \) from the relationship in Equation 3 into Equation 4 gives

\[ k \int_0^T f(t) \, dt = V_T - V_T \exp(\beta_1 f(T)^{\beta_2/2} D^\beta_3). \]  
(5)

Differentiating Equation 5 with respect to \( T \) gives

\[ kf(T) = -V_T \frac{\beta_1 D_m^\beta_2 f(T)^{\beta_2/2 - 1}}{2} \cdot \exp(\beta_1 f(T)^{\beta_2/2} D^\beta_3) \frac{df(T)}{dT}. \]  
(6)

or, following some rearrangement,

\[ k = -V_T \frac{\beta_1 D_m^\beta_2 f(T)^{\beta_2/2 - 2}}{2} \cdot \exp(\beta_1 f(T)^{\beta_2/2} D^\beta_3) \frac{df(T)}{dT}. \]  
(7)

Equation 7 is now a separable differential equation involving \( T \) and \( f(T) \). The integration of Equation 7 gives

\[ \int_0^T k \, dT = -V_T \frac{\beta_1 D_m^\beta_2 f(T)^{\beta_2/2 - 2}}{2} \int_0^T \exp(\beta_1 f(T)^{\beta_2/2} D^\beta_3) \, df(T). \]  
(8)

From Equation 8, using substitution, let

\[ x = f(T)^{\beta_2/2}, \quad dx = \frac{\beta_2}{2} f(T)^{\beta_2/2 - 1} \, df[T]. \]  
(9)

From Equation 9, \( D_m^2 = f(T) \), thus \( x = D_m^2 \). Also, when \( f(T) = 0, x = 0 \), hence the integration constant equals zero. Substituting the results from Equation 9 into Equation 8 gives

\[ kT = -V_T \frac{\beta_1 D_m^\beta_2 f(T)^{\beta_2/2 - 1}}{2} \int_0^T x^{-\beta_2/2} \exp(\beta_1 x D^\beta_3) \frac{2}{\beta_2} x^{2\beta_2/2 - 1} \, dx \]  
(10)

\[ = -V_T \frac{\beta_1 D_m^\beta_2 f(T)^{\beta_2/2 - 1}}{2} \int_0^T x^{-\beta_2/2} \exp(\beta_1 x D^\beta_3) \, dx. \]  
(11)

From Equation 10, let

\[ z = -\beta_1 x D^\beta_3, \quad dz = -\beta_1 D^\beta_3 \, dx, \]  
(12)

\[ x = \frac{z}{-\beta_1 D^\beta_3}, \quad \frac{dz}{-\beta_1 D^\beta_3} = dx. \]  
(13)

From Equation 11, \( x = D_m^2 \), thus \( z = -\beta_1 D_m^\beta_2 D^\beta_3 \), and when \( x = 0, z = 0 \), hence the integration constant equals zero.
Substitution of the results from Equation 11 into Equation 10 gives

\[ kT = -V_T \frac{\beta_1 D^{\beta_3}}{-\beta_1 D^{\beta_3}} \int_0^{z^{2/\beta_1}} \frac{e^{-z}}{(-\beta_1 D^{\beta_3})^{2/\beta_1}} \, dz, \]

\[ = V_T (-\beta_1 D^{\beta_3})^{2/\beta_1} \int_0^{z^{2/\beta_1}} e^{-z} \, dz, \]

\[ = V_T (-\beta_1 D^{\beta_3})^{2/\beta_1} \left[ \frac{e^{-z}}{-\beta_1 D^{\beta_3}} \right]_0^{z^{2/\beta_1}} + \frac{kH}{\Gamma(-\beta_1 D^{\beta_3})^{2/\beta_1}} \left[ \frac{(-\beta_1 D^{\beta_3})^{2/\beta_1}}{(-\beta_1 D^{\beta_3})^{2/\beta_1}} \right]. \]

Equation 15 gives

\[ V_C = \frac{kH}{\Gamma(-\beta_1 D^{\beta_3})^{2/\beta_1}} \left[ \frac{(-\beta_1 D^{\beta_3})^{2/\beta_1}}{(-\beta_1 D^{\beta_3})^{2/\beta_1}} \right]. \]

Table 2. Parameter estimates, standard errors and P-values from the COMP (Equations 17, 18, 19) and ground line diameter (Equation 20) system fit.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 )</td>
<td>-0.9994</td>
<td>0.0444</td>
<td>0.0001</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>4.0419</td>
<td>0.0076</td>
<td>0.0001</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>-3.9200</td>
<td>0.0151</td>
<td>0.0001</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-4.5683</td>
<td>0.4948</td>
<td>0.0001</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>1.3274</td>
<td>0.0175</td>
<td>0.0001</td>
</tr>
<tr>
<td>( \alpha_3 )</td>
<td>0.3167</td>
<td>0.0301</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Substitution of the results from Equation 11 into Equation 13 gives

\[ D_m = \Gamma^{-1} \left( \frac{k(H - M)}{\Gamma([\beta_2 - 2]/\beta_2) V_T (-\beta_1 D^{\beta_3})^{2/\beta_1}} \left( \frac{\beta_2 - 2}{\beta_2} \right)^{-1/\beta_1} \right) \]

\[ = \left( \frac{1}{-\beta_1 D^{\beta_3}} \right)^{1/\beta_1} + \varepsilon. \]

A compatible total volume (m³) outside bark equation, denoted as \( V_C \), can be readily obtained by integrating the taper function (Equation 14). However, an equivalent and simpler solution is to solve Equation 13 for \( V_T \). Solving Equation 13 for \( V_T \) and substituting \( H \) for \( T \) gives

\[ V_C = \frac{kH}{\Gamma(-\beta_1 D^{\beta_3})^{2/\beta_1}} \left[ \frac{(-\beta_1 D^{\beta_3})^{2/\beta_1}}{(-\beta_1 D^{\beta_3})^{2/\beta_1}} \right]. \]

In our case, we need to constrain Equation 15 to go from the tip to groundline diameter (\( D_o \)). The compatible total volume equation now becomes a function of total height, dbh, and groundline diameter. Substituting \( D_o \) for \( D_m \) in Equation 15, the compatible volume equation takes the form

\[ V_C = \frac{kH}{\Gamma(-\beta_1 D^{\beta_3})^{2/\beta_1}} \left[ \frac{(-\beta_1 D^{\beta_3})^{2/\beta_1}}{(-\beta_1 D^{\beta_3})^{2/\beta_1}} \right]. \]

Equation 16 is a better total volume equation, or at least completely compatible with the taper model from ground to tip.

Equation 16 is also equal to the total volume equation \( (V_T) \) presented in Equation 1. Let,

\[ V_C = V_T = \phi_1 D^{2/\beta_2} H^{\beta_3}, \]

where,

\[ \phi_1 = k \left[ \frac{(\beta_2 - 2)}{\beta_2} \right] \left[ \left( -\beta_1 D^{\beta_3} D_o^{\beta_1} \right) \right]^{-1}, \]

\[ \phi_2 = -\frac{\beta_1}{\beta_2} \quad \text{and} \quad \phi_3 = 1. \]

Substitution of \( V_C \) into the taper model gives

\[ D_m = \Gamma^{-1} \left( \frac{k(H - M)}{\Gamma([\beta_2 - 2]/\beta_2) V_C (-\beta_1 D^{\beta_3})^{2/\beta_1}} \left( \frac{\beta_2 - 2}{\beta_2} \right)^{-1/\beta_1} \right) \]

\[ = \left( \frac{1}{-\beta_1 D^{\beta_3}} \right)^{1/\beta_1} + \varepsilon. \]

The taper Equation 18 is now completely compatible with the total volume Equation 17. Subsequently, the volume ratio \( (V_R) \) presented in Equation 1 is still applicable. Thus,
Data

The suitability of the derived equations were examined with data from 126 plantation-grown loblolly pine trees representing 42 stands from the Southern Atlantic Coastal Plain of the United States. The stands were located on land owned by forest products companies, and included only stands with similar silvicultural history: 1) site preparation with no herbaceous weed control; 2) no fertilization at planting except phosphorous on phosphorous-deficient sites with no herbaceous weed control; 2) no fertilization at planting except phosphorous on phosphorous-deficient sites; 3) stand density of at least 617 trees per hectare at the time of sampling.

Outside bark diameter was measured with a diameter tape at the groundline, 0.30-, 0.61-, and 1.37-m height. The trees were then felled and measured for stump height and total height, further enhancing the utility of this compatible system. The equation takes the form

\[ D_o = \alpha_1 + \alpha_2 \text{dbh} + \alpha_3 H + \epsilon. \]  

Comparison Criteria

The comparison of the models was based on four statistical indices: coefficient of determination \( R^2 \), root mean square error (RMSE), mean bias (MB), and mean absolute error. The use of NSUR offers the best unbiased estimator for \( \beta \), which has a lower variance than the estimator of the ordinary nonlinear least squares estimator of \( \beta \), because it takes into account contemporaneous correlation in the different equations. The availability of econometric software, such as SAS/ETS (SAS Institute, Inc., 1999, v8doc.sas.com/sashtml/) makes complicated statistical procedures like NSUR easily implemented. It would be unrealistic to expect that the equation errors would be uncorrelated (Borders 1989, Parresol 1999, 2001). In an attempt to simultaneously minimize the error associated with these equations, the models in this article were fit as NSUR using the SAS/ETS Model Procedure (SAS Institute, Inc.).

Analysis

Seemingly Unrelated Regression

Development of compatible taper and volume functions involves parameter estimation of the taper function, which is integrated to provide volume. This approach will minimize the error associated with diameter estimation, but does not ensure minimal error in volume estimation. A set of nonlinear equations that has contemporaneous cross-equation error correlation is known as nonlinear seemingly unrelated regression (NSUR) system. A system of this type, which contains statistical dependencies, is best fit using nonlinear joint-generalized least squares regression. At first look, the taper, total, and merchantable volume equations seem unrelated, but the equations are related through the correlation in the errors. A set of nonlinear regression functions are specified such that 1) each equation can contain its own independent variables, 2) each equation can use its own weight function (if needed), and 3) conformity (e.g., a volume function derived from a taper function) is ensured by setting constraints on the regression coefficients (i.e., parameter sharing). The structural equations for the system of nonlinear models can be specified as:

\[
\begin{align*}
    y_1 &= f_1(X_1, \beta_1) + \epsilon_1, \\
    y_2 &= f_2(X_2, \beta_2) + \epsilon_2, \\
    \vdots \\
    y_k &= f_k(X_k, \beta_k) + \epsilon_k,
\end{align*}
\]

where \( y_i \) is a vector containing the dependent variable from the \( i \)th equation, \( X_i \) is a matrix containing the independent variables from the \( i \)th equation, \( \beta_i \) is the parameter vector for the \( i \)th equation, and \( \epsilon_i \) is the random error vector for the \( i \)th equation. When the stochastic properties of the error vectors are specified, along with the coefficient restrictions, the structural equations become a statistical model for efficient parameter estimates and reliable prediction intervals. The use of NSUR offers the best unbiased estimator for \( \beta \), which has a lower variance than the estimator of the ordinary nonlinear least squares estimator of \( \beta \), because it takes into account contemporaneous correlation in the different equations.

Comparative Analysis

The comparison of the models was based on four statistical indices: coefficient of determination \( R^2 \), root mean square error (RMSE), mean bias (MB), and mean absolute error. The comparison of the models was based on four statistical indices: coefficient of determination \( R^2 \), root mean square error (RMSE), mean bias (MB), and mean absolute error.

Table 4. Cross-equation correlation matrix of residuals for the COMP (Equations 17, 18, 19) and ground line diameter (Equation 20) system fit.*

<table>
<thead>
<tr>
<th></th>
<th>( D_m )</th>
<th>( V_C )</th>
<th>( V_{DOB} )</th>
<th>( D_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_m )</td>
<td>1</td>
<td>0.4355</td>
<td>0.3100</td>
<td>0.2142</td>
</tr>
<tr>
<td>( V_C )</td>
<td>1</td>
<td>0.7879</td>
<td>0.4149</td>
<td></td>
</tr>
<tr>
<td>( V_{DOB} )</td>
<td>1</td>
<td>0.4244</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_o )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* All correlations were found to be significant at the 0.0001 level.
bias (MAB) (Loague and Green 1991, Mayer and Butler 1993). These criteria are given as

\[ R^2 = 1 - \frac{\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}, \]  

(22)

\[ \text{RMSE} = \sqrt{\frac{\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2}{n - p}}, \]  

(23)

\[ \text{MB} = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{Y}_i), \]  

(24)

\[ \text{MAB} = \frac{1}{n} \sum_{i=1}^{n} \left| Y_i - \hat{Y}_i \right|. \]  

(25)

where, \( Y_i, \hat{Y}_i, \) and \( \bar{Y} \) are the actual, predicted, and average values of the dependent variable, \( n \) is the total number of observations used to fit the model, and \( p \) is the number of parameters in the model. The models evaluated in this article were also examined visually by plotting the fitted taper and volume ratio equations.

To truly assess performance of the models fit in this study, validation of the models with an independent data set would be the most desirable approach, because quality of fit does not always ensure the quality of prediction (Huang et al. 2003, Kozak and Kozak 2003). The commonly used validation methods of data splitting and cross-validation, as shown by Kozak and Kozak (2003), do not provide any additional information on model performance compared to the statistics obtained from models fit to the entire data set. Models validated with an independent data set prove that either the data are from the same population and will perform as per se validation utilizing data splitting, or the data are from a different population entirely, in which case the models should be refit to obtain more appropriate parameter estimates. The models fit in this study were evaluated using the goodness-of-fit statistics and graphical analyses described above.

Results

Initial parameter estimates for the COMP (Equations 17, 18, 19) models can be obtained by simultaneously fitting the outside bark total and merchantable volume equations (Equation 1). The COMP and groundline diameter prediction models were fit to the complete data set. Convergence was acquired after two iterations. All parameters in the model were found to be significant at the 0.0001 level (Table 2). As seen in Table 2, all parameter estimates are logical and ensure compatibility between equations. Fit statistics from the criterion described above are presented in Table 3. All of the COMP models were found to account for at least 96% of the variation in diameter outside bark, total, and merchantable volume. The mean bias values for all of the models were found to be negative, indicating that the models are slightly over predicting, but are generally small, suggesting that the models are overall unbiased. In addition to negative mean bias values for the COMP fit, positive correlation was found to exist among the error components (Table 4). This is intuitively reasonable because, for example, if diameter outside bark is overestimated, we would expect volume to be overestimated.

The taper function defined by Equation 18 was plotted with various dbh, total tree height, and corresponding predicted groundline diameter values (Figure 1). When height above ground is equal to zero, outside bark diameter is equal to the predicted groundline diameter. Similarly, when height above ground is equal to the total height of the tree, outside bark diameter is equal to zero. The volume ratio, defined as

![Figure 1. Outside bark taper curve from the COMP system fit with predicted groundline diameter at varying dbh (cm) and total height (m) values.](image)

582 Forest Science 51(6) 2005
cumulative volume divided by total volume, is plotted in Figure 2 with various dbh, total tree height, and corresponding predicted groundline diameter values. When height above ground is equal to total height and zero, the volume ratios are equal to one and zero, respectively.

**Discussion**

A logical, consistent, and parsimonious system of compatible taper, total, and merchantable volume outside bark equations were developed from Clark and Saucier’s (1990) volume ratio models. The equations described herein were fit simultaneously to account for cross-equation correlation among the error components. We apply this taper model to the tree stem from the top down and constrain it to predict the same total stem volume when integrated as a direct volume prediction method for the total stem. This constraint was imposed by defining the limits of integration of the taper and total volume equations. The imposed restriction ensures diameter outside bark is equal to groundline diameter and zero when height above ground is equal to zero and total tree height, respectively. Similarly, the merchantable volume equation will yield volume ratio values of zero and one when height above ground is equal to zero and total height, respectively, ensuring compatibility between the merchantable and total volume equations.

The models presented herein were found to account for a significant amount of the variation in taper, total, and merchantable volume. Here, we reparameterize our total volume equation to share the same parameters as the volume ratio. McTague and Bailey (1987) state that there is no reason why the parameters of total volume and merchantable volume equations should be estimated separately, stating that total volume may be regarded as a special case of merchantable volume when the upper stem diameter is zero. Even though McTague and Bailey (1987) simultaneously estimated the parameters of a total and merchantable volume equation, the parameter estimates for the total and merchantable parts of the model differed. The simultaneous system of equations presented in this study is unique in that the taper, total, and merchantable volume equations are composed of a single common vector of three parameter estimates, ensuring parsimony and complete compatibility.

As pointed out by one reviewer, the data used in this study were collected from trees in stands with similar growing conditions, ages, and size characteristics, from within a narrow geographical range, severely restricting the use of the model. Thus, caution should be used when extrapolating beyond the natural range of the data on which the model is based. However, the original purpose of this study was derivation of a completely compatible system of equations. The models presented in this article provide alternatives from which data collected in other regions, or for a larger more inclusive data set could be based.

**Literature Cited**


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