The Global Attractivity of a Higher Order
Rational Difference Equation

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Abstract

This paper studies global asymptotic stability for positive solutions to the equation

\[
y_n = \frac{y_{n-k}y_{n-l}y_{n-m} + y_{n-k} + y_{n-l} + y_{n-m}}{1 + y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m}}, \quad n = 0, 1, \ldots,
\]

with \(y_{-m}, y_{-m+1}, \ldots, y_{-1} \in (0, \infty)\) and \(1 \leq k < l < m\). The paper also includes a listing of possible semi-cycle structures for various \((k, l, m)\). The results generalize several others in the recent literature.

Key words: Rational Difference equation, Stability, Symmetry

2000 MSC: 39A10, 39A11

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1 Introduction

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(1)

with \( y_m, y_{m+1}, \ldots, y_{-1} \in (0, \infty) \) and \( 1 \leq k < l < m \).

The study of properties of rational difference equations has been an area of intense interest in recent years c.f. [1], [2] and the references therein.

Here we prove the following result for higher order rational equations.

**Theorem 1** Suppose that that \( \{y_i\} \) satisfies (1) with \( y_m, y_{m+1}, \ldots, y_{-1} \in (0, \infty) \). Then, the sequence \( \{y_i\} \) converges to the unique equilibrium 1.

Investigation of Equation (1) is motivated by several recent results. In particular in [4] and [5], Li investigates the qualitative behavior of the equations

\[ x_n = \frac{x_{n-1}x_{n-2}x_{n-4} + x_{n-1} + x_{n-2} + x_{n-4}}{x_{n-1}x_{n-2} + x_{n-1}x_{n-4} + x_{n-2}x_{n-4} + 1}, \quad n \in \mathbb{N}_0 \]  

(2)

and

\[ x_n = \frac{x_{n-2}x_{n-3}x_{n-4} + x_{n-2} + x_{n-3} + x_{n-4}}{x_{n-2}x_{n-3} + x_{n-2}x_{n-4} + x_{n-3}x_{n-4} + 1}, \quad n \in \mathbb{N}_0 \]  

(3)

and verifies that the positive equilibrium point of each equation is globally asymptotically stable.

We remark that stability for equations of the form

\[ y_n = \frac{1 + y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m}}{y_{n-k}y_{n-l}y_{n-m} + y_{n-k} + y_{n-l} + y_{n-m}}, \quad n = 0, 1, \ldots, \]

can also be shown via almost identical calculations to those included here.

**Remark.** It is worthwhile to note at this point that global asymptotic stability for the special cases in Equations (2) and (3) is proved in [4,5] via analysis of semi-cycle structure (similar methods are also used in [3]). Such analysis while computationally feasible for small \( m, l \) and \( k \) (see Section 4, below), can be very involved for larger values. In fact determination of semi-cycle structure as a function of \((m, l, k)\), appears to be an interesting algebraic/number theoretic problem in its own right. It is fortunate that the transformation method used here does not require prior determination of detailed semi-cycle structure.
The paper proceeds as follows. In Section 2, we introduce some preliminary lemmas and notation. Section 3 contains a proof of Theorem 1, while in Section 4, we discuss semi-cycle structure for a selection of small \( k, l \) and \( m \).

2 Preliminaries and Notation

In this section, we introduce some preliminary lemmas and notation.

First, consider the simple transformed sequence \( \{y_i^*\} \) defined by

\[
y_i^* = \begin{cases} 
y_i, & \text{if } y_i \geq 1 \\
\frac{1}{y_i}, & \text{otherwise}
\end{cases}
\]

(4)

The following elementary lemmas will be useful.

**Lemma 1** Suppose \( f \) is defined by

\[
f(x, y, z) = \frac{xyz + x + y + z}{1 + xy + xz + yz}.
\]

(5)

Then, \( f \) is decreasing in \( x \) if and only if \((y - 1)(z - 1) < 0\) and increasing in \( x \) if and only if \((y - 1)(z - 1) > 0\).

**Proof.** This follows directly from the fact that

\[
\frac{\partial}{\partial x} f(x, y, z) = \frac{(y^2 - 1)(z^2 - 1)}{(1 + xy + xz + yz)^2}.
\]

(6)

\[\square\]

**Lemma 2** Suppose that \( \{y_i\} \) satisfies (1), and that \( \{y_i^*\} \) is obtained from \( \{y_i\} \) via (4). Then, we have

\[
y_n^* = (f(y_{n-k}, y_{n-l}, y_{n-m}))^* = f(y_{n-k}^*, y_{n-l}^*, y_{n-m}^*), \quad n = 0, 1, \ldots
\]

(7)

**Proof.** Suppose that \( \{y_i\} \) satisfies (1), and for a given \( n \), set \( \mathcal{N}_n = \{ i \in \{k, l, m\} : y_{n-i} < 1 \} \). Multiplying the numerator and denominator in (1) by \( \prod_{i \in \mathcal{N}_n} y_{n-i}^* \), noting
that \( y_{n-i}y_{n-i}^* = 1 \) for \( i \in \mathcal{N} \), and simplifying in each of the eight possible cases of \( \mathcal{N} \) gives

\[
y_n = \begin{cases} 
f(y_{n-k}^*, y_{n-l}^*, y_{n-m}^*), & \text{if } ||\mathcal{N}_n|| \text{ is even} \\
1/f(y_{n-k}^*, y_{n-l}^*, y_{n-m}^*), & \text{if } ||\mathcal{N}_n|| \text{ is odd} 
\end{cases}
\]

(8)

where for a set \( \mathcal{S} \), by \( ||\mathcal{S}|| \), we denote the cardinality of \( \mathcal{S} \).

Now, note that

\[
y_{n-k}y_{n-l}y_{n-m} + y_{n-k}^* + y_{n-l}^* + y_{n-m}^* - (1 + y_{n-k}y_{n-l} + y_{n-m} + y_{n-k}y_{n-l} + y_{n-m}^*)
\]

\[
= (y_{n-k}^* - 1)(y_{n-l}^* - 1)(y_{n-m}^* - 1),
\]

(9)

and hence from (1) \( y_n > 1 \) if and only if \( ||\mathcal{N}_n|| \) is even. The lemma then follows directly from (8) and (4).

Next we prove a contraction lemma (similar to Lemma 1 in [6]) which will be helpful in showing convergence of solutions in the transformed space obtained through (4).

**Lemma 3** We have

\[
1 \leq y_n^* \leq \max\{y_{n-k}^*, y_{n-l}^*, y_{n-m}^*\},
\]

(10)

for all \( n \geq m \).

**Proof.** By Lemma 2, we have that

\[
y_n^* = \frac{y_{n-k}^*y_{n-l}^*y_{n-m}^* + y_{n-k}^* + y_{n-l}^* + y_{n-m}^*}{1 + y_{n-k}y_{n-l} + y_{n-m} + y_{n-k}y_{n-l} + y_{n-m}^*}, \quad n = 0, 1, \ldots,
\]

(11)

where \( y_i^* \geq 1 \) for all \( i \).

Setting \( x = \max\{y_{n-k}^*, y_{n-l}^*, y_{n-m}^*\} \), and applying Lemma 1 three times, we obtain

\[
y_n^* \leq \frac{x^3 + 3x}{3x^2 + 1},
\]

(12)

and the lemma follows. \( \Box \)

Now, set
for $n \geq m$.

The following result is a simple consequence of Lemma 3 and (13).

**Lemma 4** The sequence $\{D_i\}$ is monotonically non-increasing in $i$, for $i \geq m$.

Since $D_i \geq 1$ for $i \geq m$, Lemma 4 implies that, as $i$ tends to infinity, the sequence $\{D_i\}$ converges to some limit, say $D$, where $D \geq 1$.

We now turn to a proof of Theorem 1.

### 3 Convergence of solutions to Equation (1)

In this section, we give a short proof of Theorem 1.

**Proof of Theorem 1.** Note that it suffices to show that the transformed sequence $\{y_i^*\}$ converges to 1.

By the definition in (13), the values of $D_i$ are taken on by entries in the sequence $\{y_j^*\}$, and as well, by Lemma 3, $y_j^* \in [1, D_i]$ for $i \geq m$. Suppose $D > 1$. Then, for any $\epsilon \in (0, D)$, we can find an $N$ such that $y_N^* \in [D, D + \epsilon]$, and for $i \geq N - m$,

\[
y_i^* \in [1, D + \epsilon].
\]  

We will show that $D = 1$, and from this, (4), (13) and the definition of $D$, the result follows.

Since $y_i^* \geq 1$ for all $i$, employing Lemmas 1 and 2, gives

\[
D \leq y_n^* \leq \frac{(D + \epsilon)^3 + 3(D + \epsilon)}{3(D + \epsilon)^2 + 1} \leq x.
\]  

Hence

\[
3D(D + \epsilon)^2 + D \leq (D + \epsilon)^3 + 3(D + \epsilon)
\]

\[
2D^3 \leq 2D + (3\epsilon + \epsilon^3 - 3D^2\epsilon),
\]  

5
which implies \( D = 1 \), since \( \epsilon > 0 \) is arbitrary. □

In the next section, we consider briefly semi-cycle structure for a selection of small \( k \), \( l \) and \( m \).

4 Semi-cycle structure for small \( k \), \( l \) and \( m \)

In [4,5], the semi-cycle rules for the equations in (2) and (3) are given. In particular for \( k = 2, l = 3, m = 4 \) (disregarding the nonoscillatory cases) the rule is either \( 3^+, 1^-, 1^+, 2^+ \) or \( 3^-, 1^+, 1^-, 2^+ \) in a period, while for \( k = 1, l = 2, m = 4 \), we have either \( 3^+, 2^-, 1^+, 1^- \) or \( 2^+, 1^-, 1^+, 3^- \).

Table 1 below gives the semi-cycle rules for several other \( (k,l,m) \). The results are obtained computationally simply by considering \( \{(y_{n-m}, y_{n-m+1}, \ldots, y_n)\} \) relative to \((1,1,\ldots,1)\) and splitting the \( 2^m \) possible \( m \)-strings into distinct cycle classes.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( l )</th>
<th>( m )</th>
<th>Possible semi-cycle structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>( (2^-, 2^+), (1^-, 1^+) )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>( (3^-, 2^+, 1^-, 1^+), (3^+, 2^-, 1^+, 1^-) )</td>
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<td>2</td>
<td>5</td>
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References


