The difference equation $x_{n+1} = \alpha + \frac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}$ has solutions converging to zero

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Abstract

The aim of this note is to show that the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}, \quad n = 0, 1, \ldots,$$

where $k \in \mathbb{N}$, $c_i \geq 0, i = 0, \ldots, k - 1$, $\sum_{i=0}^{k-1} c_i = 1$, and $\alpha < -1$, has solutions which monotonically converge to zero. This result shows the existence of such solutions which was not shown in the recently accepted paper: A. E. Hamza, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, J. Math. Anal. Appl. (in press).

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1 Introduction

In [7] the author investigates the behavior of solutions of the difference equation

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\[ x_{n+1} = \alpha + \frac{x_n - 1}{x_n}, \quad n = 0, 1, \ldots, \]  
(1)

where \( \alpha < -1 \). The case \( \alpha > 0 \) was considered in [1] and a generalization of Eq. (1) was considered in [13]. Among other things it has been proved that if \( (x_n) \) is a nontrivial solution of Eq. (1) so that there is \( n_0 \in \mathbb{N} \) such that \( x_n \geq \alpha + 1 \) for \( n \geq n_0 \), then it is monotonically convergent to zero. Indeed it is clear that if \( x_n \geq \alpha + 1 \) for \( n \geq n_0 \), then \( \alpha + 1 \leq x_{n+1} = \alpha + \frac{x_n - 1}{x_n} \), for every \( n \geq n_0 - 1 \). If \( x_{n-1} > 0 \) this implies that \( 0 < x_{n_0} \leq x_{n_0-1} \), and by induction we have that \( 0 < \cdots \leq x_{n_0+1} \leq x_{n_0} \leq x_{n_0-1} \). Similarly if \( x_{n_0-1} < 0 \) then \( \alpha + 1 \leq x_{n_0-1} \leq x_{n_0} \leq \cdots < 0 \). Hence, the sequence \( (x_n) \) is convergent and since it cannot converge to \( \alpha + 1 \) its limit must be equal to zero. However, it was not shown that such solutions do exist, which is an interesting problem. Our aim in this note is to solve the problem in a more general framework.

Problems of this type have been investigated by several authors. General results which can help in proving the existence of monotone solutions were developed by S. Stević in [14], based on L. Berg’s ideas in [3] which use asymptotics. Asymptotics for solutions of difference equations have been investigated by L. Berg and S. Stević, see, for example, [2-5, 8-12] and the reference therein. The problem is solved by constructing two appropriate sequences \( y_n \) and \( z_n \) with

\[ y_n \leq x_n \leq z_n \quad \text{(2)} \]

for sufficiently large \( n \). In [2] and [3] some methods can be found for the construction of these bounds; see, also [4] and [5].

Let us describe briefly a Berg-Stević method which can be used in proving that a difference equation has monotone solutions. For a difference equation with the equilibrium \( \bar{x} \) we consider its linearized equation about the equilibrium. If the characteristic equation of the linearized equation has a zero \( \lambda \in (0, 1) \) we can assume that the difference equation has solutions with the following asymptotics

\[ \bar{x} + \lambda^n + o(\lambda^2n). \quad \text{(3)} \]

At this point we require a theorem which will guaranty the existence of solutions which have the needed asymptotics. A quite general result will be proved in Section 2.
Consider now the equation
\[ x_{n+1} = \alpha + \frac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{4} \]
where \( k \in \mathbb{N}, c_i \geq 0, i = 0, \ldots, k-1, \sum_{i=0}^{k-1} c_i = 1, \) and \( \alpha < -1, \) which generalizes Eq. (1). Note that the linearized equation for Eq. (4) about the equilibrium \( \bar{x} = \alpha + 1 \) is
\[ (\alpha + 1)y_{n+1} + \sum_{i=0}^{k-1} c_i y_{n-i} - y_{n-k} = 0. \tag{5} \]

For the case \( \alpha > -1 \) the characteristic polynomial associated with Eq. (5) i.e.,
\[ P(t) = (\alpha + 1)t^{k+1} + \sum_{i=0}^{k-1} c_i t^{k-i} - 1 = 0, \tag{6} \]
has a positive root \( t_0 \) belonging to the interval \((0, 1).\) To see this, note that \( P(0) = -1 \) and \( P(1) = \alpha + 1 > 0.\)

This fact motivated us to believe that in this case there are solutions of Eq. (4) which have the following asymptotics
\[ x_n = \alpha + 1 + at_0^n + o(t_0^n), \tag{7} \]
where \( a \in \mathbb{R} \) and \( t_0 \) is the above mentioned root of polynomial (6).

The problem of the existence of nonoscillatory solutions of Eq. (4) for the case \( \alpha < -1 \) is more interesting, since \( \bar{x} = 0 \) is not an equilibrium of Eq. (4)! Hence it is difficult to guess the asymptotics of the solutions which exist. Nevertheless we expect that if such solutions exist they converge to zero geometrically, moreover we expect that for such solutions the first two members in their asymptotics are in the following form:
\[ \varphi_n = at^n + bt^{2n}, \tag{8} \]
where \( t \) is a number belonging to the interval \((0, 1).\)
For the case of Eq. (4), $t$ can be chosen in the following way. Since we consider only those solutions of Eq. (4) which are defined for all $n \in \mathbb{N}$, we can write the equation in the following form

\[
(x_{n+1} - \alpha) \sum_{i=0}^{k-1} c_i x_{n-i} - x_{n-k} = 0.
\]  

(9)

For this equation, $\bar{x} = 0$ is an equilibrium and the corresponding linearized equation about the equilibrium $\bar{x} = 0$ is

\[
\alpha \sum_{i=0}^{k-1} c_i x_{n-i} + x_{n-k} = 0.
\]  

(10)

Note that the characteristic equation

\[
P_1(t) = \alpha \sum_{i=0}^{k-1} c_i t^{k-i} + 1
\]

satisfies $P_1(0) = 1$ and $P_1(1) = 1 + \alpha < 0$ and that $P_1$ is decreasing on $(0, 1)$. Hence $P_1$ has a unique characteristic zero $t = t_1$, which belongs to the interval $(0, 1)$ if $\alpha < -1$. For such chosen $t$ we expect that Eq. (4) has solutions which have the first two members in their asymptotics as in (8).

2 The inclusion theorem

The following result plays a crucial part in proving the main result. The proof of the result is similar to that of Theorem 1 in [14]. We will give a proof for the benefit of the reader.

**Theorem 1.** Let $f : I^{k+2} \to I$ be a continuous and non-decreasing function in each argument on the interval $I \subset \mathbb{R}$, and let $(y_n)$ and $(z_n)$ be sequences with $y_n < z_n$ for $n \geq n_0$ and such that

\[
y_{n-k} \leq f(n, y_{n-k+1}, \ldots, y_n), \quad f(n, z_{n-k+1}, \ldots, z_n) \leq z_{n-k},
\]

(11)
for \( n > n_0 + k - 1 \).

Then there is a solution of the following difference equation

\[
x_{n-k} = f(n, x_{n-k+1}, \ldots, x_{n+1}),
\]

(12)

with property (2) for \( n \geq n_0 \).

Proof. Let \( N \) be an arbitrary integer such that \( N > n_0 + k - 1 \). The solution \( (x_n) \) of (12) with given initial values \( x_N, x_{N+1}, \ldots, x_{N+k} \) satisfying condition (2) for \( n \in \{N, N+1, \ldots, N+k\} \) can be continued by (12) to all \( n < N \). Inequalities (11) and the monotonic character of \( f \) imply that (2) holds for all \( n \in \{n_0, \ldots, N+k\} \). Let \( A_N \) be the set of all \((k+1)\)-tuples \( (x_{n_0}, \ldots, x_{n_0+k}) \) such that there exist solutions \( (x_n) \) of (12) with these initial values satisfying (2) for all \( n \in \{n_0, \ldots, N+k\} \). It is clear that \( A_N \) is a closed nonempty set for every \( N > n_0 + k - 1 \), and that \( A_{N+1} \subset A_N \). It follows that the set \( A = \cap_{N=n_0+k}^\infty A_N \) is a nonempty subset of \( \mathbb{R}^{k+1} \) and that if \( (x_{n_0}, \ldots, x_{n_0+k}) \in A \), then the corresponding solutions of (12) satisfy (2) for all \( n \geq n_0 \), as desired.

3 The main result

In this section we prove the main result in this note.

**Theorem 2.** For each \( \alpha < -1 \) there is a nonoscillatory solution of Eq. (4) converging to zero as \( n \to \infty \).

Proof. First note that Eq. (4) can be written in the following equivalent form

\[
F(x_{n-k}, x_{n-k+1}, \ldots, x_n, x_{n+1}) \overset{\text{def}}{=}(x_{n+1} - \alpha) \sum_{i=0}^{k-1} c_i x_{n-i} - x_{n-k} = 0.
\]

(13)

We expect that solutions of Eq. (4) converging to zero have the asymptotic approximation (8). Consider

\[
F = F(\varphi_{n-k}, \ldots, \varphi_n, \varphi_{n+1}).
\]
Then, we have

\[ F = (at^{n+1} + bt^{n+2} - \alpha) \left( a \sum_{i=0}^{k-1} c_i t^{n-i} + b \sum_{i=0}^{k-1} c_i t^{2n-2i} \right) - (at^{n-k} + bt^{2n-2k}) \]

\[ = -at^{n-k} \left( \alpha \sum_{i=0}^{k-1} c_i t^{k-i} + 1 \right) \]

\[ + t^{2n} \left( a^2 \sum_{i=0}^{k-1} c_i t^{1-i} - \alpha b \sum_{i=0}^{k-1} c_i t^{-2i} - bt^{-2k} \right) + o(t^{2n}) \]

\[ = -at^{n-k} P_1(t) + t^{2n} A(a, b, t) + o(t^{2n}), \] (14)

where

\[ A(a, b, t) = a^2 \sum_{i=0}^{k-1} c_i t^{1-i} - \alpha b \sum_{i=0}^{k-1} c_i t^{-2i} - bt^{-2k} \]

\[ = a^2 \sum_{i=0}^{k-1} c_i t^{1-i} - bt^{-2k} \left( \alpha \sum_{i=0}^{k-1} c_i t^{2(k-i)} + 1 \right). \] (15)

Now, as mentioned earlier, there exists a unique \( t = t_0 \in (0, 1) \), satisfying \( P_1(t) = \alpha \sum_{i=0}^{k-1} c_i t^{k-i} + 1 = 0 \). For such chosen \( t \) choose \( b \in \mathbb{R} \), and \( a \neq 0 \), such that \( A(a, b, t_0) = 0 \). Since,

\[ \alpha \sum_{i=0}^{k-1} c_i t_0^{2(k-i)} + 1 > \alpha \sum_{i=0}^{k-1} c_i t_0^{k-i} + 1 = P_1(t_0) = 0, \] (16)

we have from (15) that

\[ b = \frac{a^2 \sum_{i=0}^{k-1} c_i t_0^{1-i}}{t_0^{-2k} \left( \alpha \sum_{i=0}^{k-1} c_i t_0^{2(k-i)} + 1 \right)} > 0. \] (17)

If \( \hat{\varphi}_n = at_0^n + qt_0^{2n} \), we obtain

\[ F(\hat{\varphi}_{n-k}, \ldots, \hat{\varphi}_n, \hat{\varphi}_{n+1}) \sim A(a, q, t_0) t_0^{2n}. \]

Since the function \( A(a, b, t_0) \) is linear in variable \( b \) and decreasing (in view of (15) and (16)), we have that there are \( q_1 < b \) and \( q_2 > b \) such that \( A(a, q_1, t_0) > 0 \) and \( A(a, q_2, t_0) < 0 \).
With the notations
\[ y_n = at_0^n + qt_0^{2n}, \quad z_n = at_0^n + q_2t_0^{2n} \]
we get
\[ F(y_{n-k}, \ldots, y_n, y_{n+1}) \sim A(a, q_1, t_0)t_0^{2n} > 0 \]
and
\[ F(z_{n-k}, \ldots, z_n, z_{n+1}) \sim A(a, q_2, t_0)t_0^{2n} < 0. \]

From this we have that inequalities (11) are satisfied for sufficiently large \( n \), where
\( f = F + x_{n-k} \) and \( F \) is given by (13). Since for \( a > 0 \) and sufficiently large \( n \), \( y_n > 0 \)
and the function \( f \) is continuous and nondecreasing in each variable, we can apply
Theorem 1 with \( I = (0, \infty) \), and see that there is an \( n_0 \geq 0 \) and a solution of Eq. (4)
with the asymptotics
\[ x_n = \hat{\phi}_n + o(t_0^{2n}), \quad \text{for} \quad n \geq n_0, \]
where \( q = b \) in \( \hat{\phi}_n \); in particular, the solution converges monotonically to zero for
\( n \geq n_0 \). Hence, the solution \( x_{n+n_0+k} \) is also such a solution when \( n \geq -k \).

**Remark 1.** Since \( a > 0 \) is an arbitrary parameter, by Theorem 2 we find a set of
solutions of Eq. (4) converging nonincreasingly to zero.

**Remark 2.** If we take \( a < 0 \) in the proof of Theorem 2 we obtain the set of solutions
of Eq. (4) converging nondecreasingly to zero.

**Remark 3.** From the proof of Theorem 2, we see that that the parameter \( \alpha \) can be
replaced by a nonincreasing sequence with the following asymptotics \( \alpha_n = \alpha + o(t_0^{2n}) \),
and that in this case there is a positive solution of the corresponding equation which
is eventually monotonic and converges to zero.

**References**


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