A NOTE ON POSITIVE NONOSCILLATORY SOLUTIONS OF
THE DIFFERENCE EQUATION $x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$

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ABSTRACT. The aim of this note is to show that the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}, \quad n = 0, 1, \ldots,$$

where $\alpha > -1$, $p > 0$ and $k \in \mathbb{N}$ is fixed, has positive nonoscillatory solutions which converge to the positive equilibrium $\bar{x} = \alpha + 1$. This result solves Open Problem 1 in S. Stević, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$, J. Appl. Math. & Computing 18 (1-2) (2005), 229-234, as well as Open Problem 1 in R. DeVault, C. Kent and W. Kosmala, On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, J. Differ. Equations Appl. 9 (8) (2003), 721-730. It is interesting that the method described here, in some cases can be applied also when the parameter $\alpha$ is variable.

1. INTRODUCTION

In [11] we investigate the behavior of the positive solutions of the difference equation

$$(1) \quad x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}, \quad n = 0, 1, \ldots,$$

where $\alpha, p > 0$ and $k \in \mathbb{N}$ is fixed. The case $p = 1$ was considered in [5] DeVault, Kent and Kosmala. Among other things it has been proved that all nonoscillatory solutions of Eq.(1) converge to the positive equilibrium $\bar{x} = \alpha + 1$. Indeed it is clear that if $x_n \geq \bar{x}$ for $n \geq -k$, then $\alpha + 1 \leq x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$, which implies that $\alpha + 1 \leq x_n \leq x_{n-k}$, for $n = 0, 1, \ldots$. Hence $\lim_{m \to \infty} x_{mk+i}$ exists for each $i \in \{0, 1, \ldots, k-1\}$. From this and (1) the result follows. The case $x_n \leq \bar{x}$, can be treated similarly.

Based on this observation S. Stević ([11]), and for the case $p = 1$, DeVault, Kent and Kosmala ([5]) have posed the following open problem:

Open Problem. Do there exist nonoscillatory solutions of Eq.(1)?

Our aim in this note is to solve the open problem. Moreover, our proof holds also for the case $\alpha \in (-1, 0]$.

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Note that the linearized equation for Eq. (1) about the positive equilibrium $\bar{x}$ is
\[(\alpha + 1)y_{n+1} + py_n - py_{n-k} = 0.\]

The characteristic polynomial associated with Eq. (2) is
\[P(t) = (\alpha + 1)t^{k+1} + pt^k - p = 0.\]

Since $P(0) = -p < 0$, $P(1) = \alpha + 1$ and $P'(t) = (\alpha + 1)(k + 1)t^k + pkt^{k-1} > 0$, when $t \in (0, 1)$, it follows that for each $\alpha > -1$, there is a unique positive root $t_0$ of the polynomial belonging to the interval $(0, 1)$.

This fact motivated us to believe that there are solutions of Eq. (1) which have the following asymptotics
\[x_n = \bar{x} + at_0^n + o(t_0^n),\]
where $a \in \mathbb{R}$ and $t_0$ is the above mentioned root of polynomial (3).

We solve the open problem, showing that such a solution exists, developing Berg’s ideas in [2] which are based on asymptotics. Asymptotics for solutions of difference equations have been investigated by L. Berg and S. Stević, see, for example, [1-4,6-10] and the reference therein. The problem is solved by constructing two appropriate sequences $y_n$ and $z_n$ with
\[y_n \leq x_n \leq z_n\]
for sufficiently large $n$. In [1] and [2] some methods can be found for the construction of these bounds; see, also [3] and [4].

From (4) and results in Berg’s paper [2] we expect that for $k \geq 2$ such solutions have the first three members in their asymptotics in the following form:
\[\varphi_n = \bar{x} + at^n + bt^{2n}.\]

2. THE INCLUSION THEOREM

We need the following result in the proof of the main theorem. The proof of the result is similar to that of Theorem 1 in [2].

**Theorem 1.** Let $f : I^{k+2} \rightarrow I$ be a continuous and non-decreasing function in each argument on the interval $I \subset \mathbb{R}$, and let $(y_n)$ and $(z_n)$ be sequences with $y_n \leq z_n$ for $n \geq n_0$ and such that
\[y_{n-k} \leq f(n, y_{n-k+1}, \ldots, y_{n+1}), \quad f(n, z_{n-k+1}, \ldots, z_{n+1}) \leq z_{n-k},\]
for $n > n_0 + k - 1$.

Then there is a solution of the following difference equation
\[x_{n-k} = f(n, x_{n-k+1}, \ldots, x_{n+1}),\]
with property (5) for $n \geq n_0$.

**Proof.** Let $N$ be an arbitrary integer such that $N > n_0 + k - 1$. The solution $(x_n)$ of (8) with given initial values $x_N, x_{N+1}, \ldots, x_{N+k}$ satisfying condition (5) for $n \in \{N, N+1, \ldots, N+k\}$ can be continued by (8) to all $n < N$. Inequalities (7) and the monotonic character of $f$ imply that (5) holds for all $n \in \{n_0, \ldots, N + k\}$. Let $A_N$ be the set of all $(k + 1)$-tuples $(x_{n_0}, \ldots, x_{n_0+k})$ such that there exist solutions $(x_n)$ of (8) with these initial values satisfying (5) for all $n \in \{n_0, \ldots, N + k\}$. It is clear that $A_N$ is a closed nonempty set for every $N > n_0 + k - 1$, and that $A_{N+1} \subset A_N$. It follows that the set $A = \cap_{N=n_0+k}^\infty A_N$ is a nonempty subset of $\mathbb{R}^{k+1}$.
and that if \((x_{n_0}, \ldots, x_{n_0+k}) \in A\), then the corresponding solutions of (8) satisfy (5) for all \(n \geq n_0\), as desired.

3. THE MAIN RESULT

In this section we prove the main result in this note.

**Theorem 2.** For each \(\alpha > -1\) there is a nonoscillatory solution of Eq.(1) converging to the positive equilibrium \(x = \alpha + 1\), as \(n \to \infty\).

**Proof.** First note that Eq.(1) can be written in the following equivalent form

\[
F(x_{n-k}, x_n, x_{n+1}) = (x_{n+1} - \alpha)^{1/p} x_n - x_{n-k} = 0.
\]

We expect that solutions of Eq.(1) have the asymptotic approximation (6). Thus, we calculate \(F(\varphi_{n-k}, \varphi_n, \varphi_{n+1})\). We have

\[
F = \frac{a}{p} t^n (\alpha + 1) t + p - \frac{p}{t^k}
\]

\[
+ \frac{1}{p} b (\alpha + 1) t^2 + \frac{b p}{t^k} + a^2 t + \frac{(1-p) a^2 (\alpha + 1)}{2p} t^2) + o(t^{2n}).
\]

Let \(D(t) = (\alpha + 1) t + p - pt^{-k}\). Choose \(t \in (0, 1)\) such that \(D(t) = 0\), and \(a, b \in \mathbb{R}\), \(\alpha \neq 0\), such that the coefficients in Eq.(10) are equal to zero. \(D(t) = 0\) implies that \(t = t_0\) (see the introduction). From this and (10), we have that

\[
b = -\frac{a^2 t_0 + \frac{(1-p) a^2 (\alpha + 1)}{2p} t_0^2}{(\alpha + 1) t_0^2 + p - pt_0^{-2k}} = -\frac{a^2 t_0 + \frac{(1-p) a^2 (\alpha + 1)}{2p} t_0^2}{D(t_0^2)}.
\]

Observe that since \(D'(t) = \alpha + 1 + \frac{bp}{t^{k+1}} > 0\) when \(t \in (0, 1)\), then \(D(t_0^2) < D(t_0) = 0\). If \(\varphi_n = \alpha + 1 + at_0^n + qt_0^{2n}\), we obtain

\[
F(\varphi_{n-k}, \varphi_n, \varphi_{n+1}) \approx \frac{1}{p} \left( qD(t_0^2) + a^2 t_0 + \frac{(1-p) a^2 (\alpha + 1)}{2p} t_0^2 \right) t_0^{2n}.
\]

Let

\[
H_t_0(q) = qD(t_0^2) + a^2 t_0 + \frac{(1-p) a^2 (\alpha + 1)}{2p} t_0^2.
\]

Since \(H_t_0(q_1) = D(t_0^2) < 0\), we obtain that there are \(q_1 < b\) and \(q_2 > b\) such that \(H_t_0(q_1) > 0\) and \(H_t_0(q_2) < 0\).

With the notations

\[
y_n = \bar{x} + at_0^n + q_1 t_0^{2n}, \quad z_n = \bar{x} + at_0^n + q_2 t_0^{2n}
\]

we get

\[
F(y_{n-k}, y_n, y_{n+1}) \sim \frac{1}{p} \left( q_1 D(t_0^2) + a^2 t_0 + \frac{(1-p) a^2 (\alpha + 1)}{2p} t_0^2 \right) t_0^{2n} > 0
\]

and

\[
F(z_{n-k}, z_n, z_{n+1}) \sim \frac{1}{p} \left( q_2 D(t_0^2) + a^2 t_0 + \frac{(1-p) a^2 (\alpha + 1)}{2p} t_0^2 \right) t_0^{2n} < 0.
\]

These relations show that inequalities (7) are satisfied for sufficiently large \(n\), where \(f = F + x_{n-k}\) and \(F\) is given by (9). Since for sufficiently large \(n\), \(y_n > \alpha\), we can apply Theorem 1 with \(I = (\alpha, \infty)\), and see that there is an \(n_0 \geq 0\) and a solution
of Eq. (1) with the asymptotics $x_n = \hat{x}_n + o(t_0^n)$, for $n \geq n_0$, where $q = b$ in

$\hat{x}_n$; in particular, the solution converge monotonically to the positive equilibrium

$\hat{x} = \alpha + 1$, for $n \geq n_0$. Hence, the solution $x_{n+n_0+k}$ is also such a solution when

$n \geq -k$.

**Remark 1.** Since $a \in \mathbb{R} \setminus \{0\}$ is an arbitrary parameter, by Theorem 2 we find a
set of nonoscillatory solutions of Eq. (1) converging to the positive equilibrium.

**Remark 2.** From the proof of Theorem 2, we see that the parameter $\alpha$ can be replaced by a nonincreasing sequence with the following asymptotics $\alpha_n = \alpha + o(t_0^n)$, and that in this case there is a positive solution of the corresponding
equation which is eventually nonoscillatory.

**References**


No.4, (2002), 1061-1074.

Equations Appl. 10 (4) (2004), 399-408.

[4] L. Berg, Corrections to “Inclusion theorems for non-linear difference equations with appli-
cations,” from [3], J. Differ. Equations Appl. 11 (2) (2005), 181-182.

[5] R. DeVault, C. Kent and W. Kosmala, On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, J.

(1996), 99-105.


[10] S. Stević, On the recursive sequence $x_{n+1} = x_n + \frac{c}{x_n}$, Bull. Calcuta Math. Soc. 95 (1)
(2003), 39-46.

[11] S. Stević, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, J. Appl. Math & Computing 18
(1-2) (2005), 229-234.