BOUND edness Character of POSITIVE SOLUTIONS OF A Max Difference Equation

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Abstract. This paper studies the boundedness character of the positive solutions of the difference equation

\[ y_n = \max \left\{ c, \frac{y_{n-k}}{y_{n-m}} \right\}, \quad n \in \mathbb{N}_0, \]

where \( k,m \in \mathbb{N} \) with \( \gcd(k,m) = 1 \). We prove that if \( c \geq 1 \), then every solution of the equation is bounded, and if \( c \in (0, 1) \) and \( k \) is even, then there exist positive unbounded solutions. For the case \( c \in (0, 1) \) and \( k \) odd, we consider the related equation \( y_n = \max \left\{ c, y_{n-k} - y_{n-m} \right\} \) and show that every integer solution is eventually periodic.

1. Introduction

The investigation of the difference equation

\[ x_n = \max \left\{ \frac{A_1}{x_{n-1}}, \frac{A_2}{x_{n-2}}, \ldots, \frac{A_r}{x_{n-r}} \right\}, \quad n \in \mathbb{N}_0 \]

where \( A_i, i = 1, \ldots, r, \) are real numbers, such that at least one of them is different from zero and the initial conditions \( x_{-1}, \ldots, x_{-r} \), are different from zero, was proposed by Ladas in [2] and [3]. The max type operators arise naturally in certain models in automatic control theory (see [5, 6]). The case \( r = 2 \) was thoroughly investigated in [1].

The equation

\[ x_n = \max \left\{ \frac{A}{x_{n-1}}, \frac{B}{x_{n-3}} \right\} \]

where \( A \) and \( B \) are any positive coefficients was investigated in [4]. Using the change \( y_n = x_{n+1}x_n/B \) Eq. (2) becomes

\[ y_n = \max \left\{ D, \frac{y_{n-1}}{y_{n-2}} \right\} \]

where \( D = AB^{-1} > 0 \).

Among other results in [4], the authors prove the following theorem.
**Theorem 1.** Let \( \{y_n\} \) be a solution of Eq. (3) such that \( y_{-2} > 0, \, y_{-1} > 0 \), then

\[
D \leq y_n \leq \max \left\{ D, \frac{1}{D} \right\}, \quad \text{for } n \geq 5.
\]

This result motivated us to investigate the boundedness of positive solutions of the following generalization of Eq. (3):

\[
y_n = \max \left\{ c, \frac{y_{n-k}}{y_{n-m}} \right\}, \quad n \in \mathbb{N}_0,
\]

where \( k, m \in \mathbb{N} \).

In [7] Stević, posed the following problem.

**Problem.** Investigate the boundedness character of positive solutions of Eq. (5).

**Remark 1.** Note that the case \( k = m \) is trivial, since in this case \( y_n = \max\{c, 1\} \), for \( n \geq 0 \).

As per usual if \( g = \gcd(k, m) > 1 \), then \( \{y_i\} \) satisfying (5) can be separated into \( g \) different equations of the form

\[
y^{(j)}_n = \max \left\{ c, \frac{y^{(j)}_{n-k}}{y^{(j)}_{n-m}} \right\},
\]

where \( j \in \{1, 2, \ldots, g\} \). Hence, we will restrict attention to the case \( \gcd(k, m) = 1 \).

The remainder of the paper proceeds as follows. In Section 2, we give some generalizations of Theorem 1, and in particular, prove boundedness of solutions to (5) for the case \( c \geq 1 \). Sections 3 and 4 provide results for the cases \( k \) even and \( k \) odd, respectively.

2. **Boundedness of the positive solutions of Eq. (5)**

In this section we consider the boundedness of positive solutions of an extension of Eq. (1), as well as some closely related results.

First, suppose that \( c_i > 0 \) for \( i \geq 0 \), and consider the general equation

\[
y_n = \max \left\{ c_n, \frac{y_{n-k}}{y_{n-m}} \right\}, \quad n = 0, 1, \ldots,
\]
with $y_{-s}, y_{-s+1}, \ldots, y_{-1} > 0$ and $k, m \in \{1, 2, 3, 4, \ldots\}$, where $s = \text{max}\{k, m\}$. Note that iteratively applying (7) to $y_{n-i}k$, $0 \leq i \leq T-1$, gives that $\{y_i\}$ satisfies

\[ y_n = \max \left\{ c_n, \frac{c_{n-k}}{y_{n-m}}, \frac{c_{n-2k}}{y_{n-m}y_{n-k-m}}, \ldots, \frac{c_{n-(T-1)k}}{\prod_{i=0}^{T-2} y_{n-ik-m}}, \frac{y_{n-Tk}}{\prod_{i=0}^{T-1} y_{n-ik-m}} \right\}, \]

for $n \geq (T-1)k$.

The following result is closely related to Theorem 2 in [8].

**Theorem 2.** Suppose that $k = 1$, and

\[ l \leq c_i \leq M \]

for all $i \geq 0$. Then,

\[ l \leq y_n \leq \max \left\{ M, \frac{M}{l^{m-1}}, \frac{M}{l^{m-1}}, \frac{1}{l^{m-1}} \right\}, \]

for $n \geq 2m$.

**Proof.** Since $k = 1$, applying (8), we have for $T = m$ and $n \geq 2m$, that

\[ y_n = \max \left\{ c_n, \frac{c_{n-1}}{y_{n-m}}, \frac{c_{n-2}}{y_{n-m}y_{n-1-m}}, \ldots, \frac{c_{n-(m-1)}}{\prod_{i=0}^{m-2} y_{n-i-m}}, \frac{y_{n-m}}{\prod_{i=0}^{m-1} y_{n-i-m}} \right\}, \]

(11)

\[ = \max \left\{ c_n, \frac{c_{n-1}}{y_{n-m}}, \frac{c_{n-2}}{y_{n-m}y_{n-1-m}}, \ldots, \frac{c_{n-(m-1)}}{\prod_{i=0}^{m-2} y_{n-i-m}}, \frac{1}{\prod_{i=1}^{m-1} y_{n-i-m}} \right\}. \]

The result then follows upon applying the bound in (9) and the fact that $y_i \geq c_i$ for $i \geq 0$. \( \square \)

If $l = M$ then we have the following corollary.

**Corollary 1.** If $\{y_i\}$ satisfies (7) with $c_i \equiv c > 0$ for all $i \geq 0$ then

\[ c \leq y_n \leq \max \left\{ c, \frac{1}{c}, \ldots, \frac{1}{c^{m-1}} \right\}, \]

(12)

\[ = \max \left\{ c, \frac{1}{c^{m-1}} \right\}, \]

for $n \geq 2m$. In particular if $c \geq 1$, then $y_n \equiv c$ for all $n \geq 2m$.

Returning to Eq. (5), we have the following for the case $c \geq 1$.

**Theorem 3.** Suppose $c \geq 1$. Then every solution of (5) is bounded.
Proof. Since \( y_n \geq c \geq 1 \), from (8), we have

\[
y_n = \max \left\{ c, \frac{c}{y_{n-m}}, \frac{c}{y_{n-m}y_{n-k-m}}, \ldots, \frac{c}{\prod_{i=0}^{T-2} y_{n-ik-m}}, \frac{y_{n-Tk}}{\prod_{i=0}^{T-1} y_{n-ik-m}} \right\}
\]

for \( n \geq (T-1)k \). Setting \( T = \lceil \frac{n}{k} \rceil \), we have \( 0 \leq n - Tk \leq k - 1 \), and (13) gives

\[
y_n \leq \max\{c, y_0, y_1, \ldots, y_{k-1}\}
\]

for all \( n \geq 0 \). \( \square \)

We now turn to consideration of boundedness for positive solutions of (5), when \( k \) is even.

3. Case \( k \) even

In this section we consider Eq. (5), for the case, when \( k \) is even and \( c \in (0,1) \).

**Theorem 4.** Suppose that \( k \) is even, \( m \) is odd and \( c \in (0,1) \). Then there exists an unbounded positive solution of Eq. (5).

**Proof.** We may assume that \( k > m \); the case \( k < m \) is similar and will be omitted.

For \(-m \leq i \leq -1\), let \( y_i = c \) if \( i \) is odd and \( y_i = 1 \) if \( i \) is even. Then, for \( 0 \leq i < k \), we have \( y_i = c \) if \( i \) is odd and \( y_i = 1/c \) for \( i \) even. Hence assume

\[
y_i = \begin{cases} 
  c, & \text{if } i < Vk \text{ and } i \text{ is odd} \\
  \frac{1}{c^v}, & \text{if } (v-1)k \leq i < vk \text{ and } i \text{ is even}
\end{cases}
\]

for \( v < V \). Then, for \( Vk \leq i < (V+1)k \), we have from (15), and the fact that \( c \leq 1 \), that \( y_i = c \) if \( i \) is odd and \( y_i = 1/c^v \) if \( i \) is even. The result then follows since \( \{c^{-v}\}_{v \geq 0} \) is unbounded. \( \square \)

4. Case \( k \) odd

In this section we consider the case \( c \in (0,1) \) and \( k \) odd.

First, note that by using the change \( y_n = 1/c^z \) Eq. (5) becomes:

\[
z_n = \max\{-1, z_{n-k} - z_{n-m}\}, \quad n \in \mathbb{N}_0.
\]

Now, we will prove the following regarding asymptotic periodicity for integer solutions to Equation (16).

**Theorem 5.** Suppose that \( \{z_n\} \) satisfies (16) with \( z_{-s}, z_{-s+1}, \ldots, z_{-1} \in \mathbb{Z} \) where \( s = \max\{m, k\} \). Then, if \( k \) is odd, the sequence \( \{z_n\} \) converges to a periodic solution of the equation.
**Proof.** Note that if \( \{ z_n \} \) is bounded then it is asymptotically periodic. To see this, note that there are only finitely many different \( s \)-tuples with integer values in \([-1, A]\) for a given \( A > 0 \) and hence if \(-1 \leq z_n \leq A\) for all \( n \geq 0 \), then a string of \( s \) values must eventually reoccur in the sequence, thus creating a cycle. Hence, assume that \( \{ z_n \} \) is unbounded.

Now, we show that it is possible to find positive numbers \( N \) and \( M \) satisfying the following properties

(i) \( z_N = M \),
(ii) \( z_i < M \) for \( 0 \leq i < N \),
(iii) \( N > k \zeta > m \),
(iv) \( M > 2 \zeta \),

where \( \eta = \left\lfloor \sqrt{2M} \right\rfloor + 3 \) and \( \zeta = k \eta \). Indeed, that there are positive numbers \( N \) and \( M \) satisfying (i), (ii) and (iv) is trivial. Now we show that they could be chosen such that they satisfy condition (iii), as well.

From Eq. (16), we have that

\[
(17) \quad z_n = \max\{-1, z_{n-k} - z_{n-m}\} \leq \max\{-1, z_{n-k} + 1\} \leq z_{n-k} + 1,
\]

for \( n \geq s \), and by iterating we have that

\[
(18) \quad z_n \leq z_{n-ik} + i,
\]

for \( i \geq 1 \) satisfying \( 0 \leq \min\{n - ik, n - (i - 1)k - m\} \).

Now, note that for \( N > s \), there exists a \( v_N \) such that

\[
(19) \quad 0 \leq N - v_N k, \; N - (v_N - 1)k - m \leq s.
\]

Set \( T = \max_{0 \leq i \leq s} z_i \). From (16), (18) and (19), we have

\[
(20) \quad M = z_N \leq z_{N-v_N k} + v_N \leq T + v_N \leq T + \frac{N}{k},
\]

where the last inequality follows from the left hand inequality in (19).

Finally, for sufficiently large \( M \), (20) gives

\[
(21) \quad N \geq (M - T)k \geq k^2 \left( \left\lfloor \sqrt{2M} \right\rfloor + 3 \right) = \zeta k,
\]

as desired.

Now for convenience, we will relabel \( u_i = z_{N-k \zeta + i} \), for \( i \geq 0 \), so that \( u_0 = z_{N-k \zeta} \) and \( u_k \zeta = z_N \).
Note that \( u_{k\zeta} = M \) and via (ii) and (16), we have
\[
M = u_{k\zeta} = \max \{-1, u_{(\zeta-1)k} - u_{k\zeta-m}\} = u_{(\zeta-1)k} - u_{k\zeta-m}.
\]

From this and (ii), it follows that
\[
M - 1 \geq u_{(\zeta-1)k} = M + u_{k\zeta-m} \geq M - 1,
\]
that is \( u_{(\zeta-1)k} = M - 1 \). Iteratively, we obtain
\[
u_{ik} \geq M - \zeta + i,
\]
for \( 0 \leq i \leq \zeta \).

Next, by (ii), \( u_m \leq M - 1 \) and hence applying (24) and (iv),
\[
u_{m+k} = \max \{-1, u_m - u_k\} \leq (M - 1) - (M - \zeta + 1)
\]
\[
\leq \zeta - 2 < M - \zeta + 2 \leq u_{2k}.
\]
Thus,
\[
u_{2k+m} = \max \{-1, u_{m+k} - u_{2k}\} = -1,
\]
and iteratively,
\[
u_{ik+m} = -1,
\]
for \( 2 \leq i \leq \zeta \).

Now, from (16), we have \( u_{k+2m} \geq -1 \) and hence repeated application of (27) gives
\[
u_{ik+2m} = u_{(i-1)k+2m} + 1 \geq i - 2,
\]
for \( 1 \leq i \leq \zeta \).

Next, from (ii), \( u_{2k+3m} \leq M - 1 \) and repeated application of (28) gives
\[
u_{ik+3m} \leq \max \left\{-1, M - 1 - \frac{(i-3)(i-2)}{2}\right\},
\]
for \( 2 \leq i \leq \zeta \).

Now, note that from (29) if \( i \geq \eta = \left\lfloor \sqrt{2M} \right\rfloor + 3 \), then
\[
u_{ik+3m} = -1.
\]
Repeating the argument following (27) with (30) in place of (27) gives

\[(31)\quad u_{ik+5m} = -1,\]

for \(2\eta \leq i \leq \zeta\), and more generally,

\[(32)\quad u_{ik+(2\nu+1)m} = -1,\]

for \(1 \leq \nu \leq k-1\) and \(\nu \eta \leq i \leq \zeta\).

In particular, for \(\nu = \frac{k-1}{2}\), we have

\[(33)\quad u_{ik+km} = u_{(i+m)k} = -1,\]

for \(\nu \eta \leq i \leq \zeta\). Taking \(i = \nu \eta\) gives

\[(34)\quad u_{(\nu \eta+m)k} = -1,\]

but from (24) and (iv),

\[(35)\quad u_{(\nu \eta+m)k} = M - \zeta + \nu \eta + m > -1.\]

This contradicts (34), and the proof is complete. \(\square\)

The following result on solutions to (5) follows directly from Theorem 5.

**Theorem 6.** Suppose that \(c \in (0, 1)\) and set \(S_c = \{c^{-v}|v \in \mathbb{Z}\}\). If \(\{y_i\}\) satisfies (5) with \(k\) odd, and \(y_{-i} \in S_c\) for \(1 \leq i \leq s\), then \(\{y_i\}\) converges to a periodic solution of (5).

We conclude with the following conjecture regarding solutions to equation (16).

**Conjecture 1.** All positive solutions to Equation (16) with \(k\) odd are bounded.

**References**


[8] S. Stević, On the recursive sequence $x_{n+1} = \frac{A}{\prod_{i=0}^{k} x_{n-i}} + \frac{1}{\prod_{j=k+2}^{n+k+1} x_{n-j}}$, Taiwanese J. Math. 7 (2) (2003), 249-259.

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