EXPPLICIT BOUNDS FOR THIRD-ORDER DIFFERENCE EQUATIONS

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(Received Day Month Year; revised Day Month Year)

Abstract

This note gives explicit, applicable bounds for solutions of a wide class of third-order difference equations with nonconstant coefficients. The techniques used are readily adaptable for higher order equations. The results extend recent work of the authors for second-order equations.

Key words: Explicit bounds, Applicable bounds, Third order linear difference equations, Growth rates, Nonconstant coefficients.


1. Introduction

This paper studies explicit, applicable growth rates for third-order difference equations. In particular, we will consider solutions \(\{b_n\} = \{b_n(b_0, b_1, b_2)\}\) of equations of the form

\[
\Delta^3 b_{n-2} = p_n b_n - q_n b_{n-1} + r_n b_{n-2},
\]

where for a sequence \(\{a_i\}\), \(\Delta\) is the forward difference operator, \(\Delta a_i = a_{i+1} - a_i\). That is,

\[
b_{n+1} = (3 + p_n) b_n - (3 + q_n) b_{n-1} + (1 + r_n) b_{n-2},
\]

for \(n \geq 2\). We provide sharp inequalities for \(\{b_i\}\) in terms of the sequences \(\{p_i\}, \{q_i\},\) and \(\{r_i\},\) and the initial values \(b_0, b_1,\) and \(b_2\). Solutions of

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difference equations of the form in (1) have been studied by many authors (cf. [2]–[12]). Often the study has focused on the understanding of oscillatory or asymptotic behavior.

In what follows, it will be convenient to have the following notation. For a sequence \(a = \{a_i\}\), define the linear operator \(\mathcal{L}\) by

\[
\mathcal{L}(a)_i \overset{\text{def}}{=} p_{i+1}a_{i+1} - q_{i+1}a_i + r_{i+1}a_{i-1},
\]

for \(i \geq 1\).

We now state our main result which extends recent results for second-order equations (cf. [1] and [13]). Closely related results can also be found in [14].

**Theorem 1.** Suppose \([B_i], \{p_i\}, \{q_i\}, \text{ and } \{r_i\}\) are real-valued sequences such that \([B_i]\) is positive, non-decreasing, and convex, and for each \(i \geq 2\), either

\[
p_i \geq \max(q_i, 0) \text{ and } r_i \geq 0, \quad \text{or} \quad q_i \leq \min(r_i, 0) \text{ and } p_i \geq 0.
\]

In addition, suppose there exist positive constants, \(c_0, c_1, \text{ and } c_2\), satisfying

\[
\Delta^2 B_i \geq \begin{cases} 
  c_0\Delta^2 b_i(1, 0, 0) &= \Delta^2 b_i(c_0, 0, 0) \geq 0 \\
  c_1\Delta^2 b_i(0, -1, 0) &= \Delta^2 b_i(0, -c_1, 0) \geq 0 \\
  c_2\Delta^2 b_i(0, 0, 1) &= \Delta^2 b_i(0, 0, c_2) \geq 0
\end{cases},
\]

for \(i = 0, 1, 2\). Now, define the sequence \([V(i)]\) via

\[
V(i) \overset{\text{def}}{=} \Delta^3 B_{i-1} - \mathcal{L}(B)_i,
\]

for \(i \geq 1\). If

\[
V(i) \geq 0,
\]

for \(i \geq 3\), then

\[
|b_n| \leq \left( \frac{|b_0|}{c_0} + \frac{|b_1|}{c_1} + \frac{|b_2|}{c_2} \right) B_n,
\]

for \(n \geq 3\).

The key to employing Theorem 1 is to determine a positive, non-decreasing sequence \(B\) satisfying (5) and (7). While this can be done inductively for many \(\{(p_j, q_j, r_j)\}\), it is particularly convenient when the third derivative of an extension, \(\tilde{B}\), to \([0, \infty)\) of the bounding sequence \(B\) exists. The next lemma follows directly from the fact that \(\Delta^3 B_{n-1} = B'''(\zeta)\), for some \(\zeta \in [n-1, n+2]\).

**Lemma 1.** Suppose \(\tilde{B}'''\) exists.
1. If $\tilde{B}'''$ is non-decreasing, and for $n \geq n_0$,
\[ L(B)_n \leq \tilde{B}'''(n - 1), \] (9)
then $V(n) \geq 0$ for $n \geq n_0$.

2. If $B'''$ is non-increasing, and for $n \geq n_0$,
\[ L(B)_n \leq \tilde{B}'''(n + 2), \] (10)
then $V(n) \geq 0$ for $n \geq n_0$.

It will be helpful to have the following notation, which will be useful when demonstrating that (5) holds for particular examples.

For given \{b_i\} and \{B_i\}, define $G$ and $h$ via
\[
G = \begin{bmatrix}
g_{0,0} & g_{0,1} & g_{0,2} \\
g_{1,0} & g_{1,1} & g_{1,2} \\
g_{2,0} & g_{2,1} & g_{2,2}
\end{bmatrix}
\]
and
\[
h = (h_0, h_1, h_2) \overset{def}{=} (\Delta^2 B_0, \Delta^2 B_1, \Delta^2 B_2).
\]

Note that (5) can be rewritten as $h_i \geq c_j g_{j,i} \geq 0$, for $0 \leq i, j \leq 2$. In fact, if $h_i > 0$ and $g_{j,i} > 0$, for $0 \leq i \leq 2$, we may take $c_j = \min_{0 \leq i \leq 2} \{ \frac{h_i}{g_{j,i}} \}$.

We now give some examples of applications for Theorem 1.

**Example 1.** (Power-type rate bounds) Consider \{B_n\} defined by $B_n = n^k$ (with $k \in \mathbb{R}$), and note that $\tilde{B}$ given by $\tilde{B}(x) = x^k$, is positive, non-decreasing and convex for $k \geq 1$. Taking derivatives gives $\tilde{B}'''(x) = k(k - 1)(k - 2)x^{k-3}$ and $\tilde{B}'''(x) = k(k - 1)(k - 2)(k - 3)x^{k-4}$, and hence, $\tilde{B}'''$ is non-decreasing for $1 \leq k \leq 2$ and $k \geq 3$, and non-increasing for $2 \leq k \leq 3$.

Now, set $c = k(k - 1)(k - 2)$. Employing Lemma 1, each of the following satisfy (7) of Theorem 1.

(i) $p \equiv q \equiv 0$, $k \geq 3$, and for $n \geq 3$,
\[
0 \leq r_{n+1} \leq \frac{c}{(n-1)^3}. \] (13)

(ii) $p \equiv q \equiv 0$, $k \in [2, 3]$, and for $n \geq 3$,
\[
0 \leq r_{n+1} \leq \left( \frac{n+2}{n-1} \right)^k \frac{c}{(n+2)^3}. \] (14)
(iii) $q \equiv r \equiv 0$, $k \geq 3$, and for $n \geq 3$,

$$0 \leq p_{n+1} \leq \frac{c(n-1)^{k-3}}{(n+1)^k}. \quad (15)$$

(iv) $q \equiv r \equiv 0$, $k \in [2, 3]$, and for $n \geq 3$,

$$0 \leq p_{n+1} \leq \frac{c(n+2)^{k-3}}{(n+1)^k}. \quad (16)$$

For $k \geq 2$, $c$ is non-negative, and hence, the sequences in (i)–(iv) all satisfy (4). Now, note that (a) $r_n$ defined by $r_n = c/n^3$ satisfies both (13) and (14), and (b) $p_n$ defined by $p_n = c/(n+1)^3$ satisfies (16). We will consider these two instances in some detail.

(a) ($r_n = c/n^3$) That $r_n = c/n^3$ satisfies (13) is immediate. To see that the right hand inequality in (14) also holds, note that

$$(n + 2)^{3-k}(n-1)^k \leq (n + 2)(n - 1)^2 = n^3 - 3n + 2 < (n + 1)^3. \quad (17)$$

Now, employing the formulae in Table 2 below, we have the values

$$G = \begin{bmatrix} 1 & 1 + \frac{c}{8} & 1 + \frac{c}{8} \\ 2 & 2 & 2 - \frac{c}{27} \\ 1 & 1 & 1 \end{bmatrix}. \quad (18)$$

Hence, there exist $c_0 > 0$, $c_1 > 0$, and $c_2 > 0$ satisfying (5), whenever $0 < c/27 < 2$, i.e. $2 < k < k_0$, where $k_0 \approx 4.867936$. For example, when $k = 3$ ($c = 6$), we have $h = (6, 12, 18)$, and

$$G = \begin{bmatrix} 1 & \frac{7}{4} & \frac{7}{4} \\ 2 & 2 & \frac{16}{9} \\ 1 & 1 & 1 \end{bmatrix}. \quad (19)$$

Thus, taking ratios as suggested earlier, we may use $c_0 = c_2 = 6$, and $c_1 = 3$ in (8).

(b) ($p_n = c/(n+1)^3$) Here we have

$$p_{n+1} = \frac{c}{(n+2)^3} \leq \frac{c}{(n+2)^3} \left(\frac{n+2}{n+1}\right)^k. \quad (20)$$
and (16) is satisfied. In addition, 
\[
G = \begin{bmatrix}
1 & 1 & 1 + p_3 \\
2 & 2 & 2 + 3p_3 \\
1 & 1 + p_2 & 1 + 3p_3 + p_2 + p_3p_2
\end{bmatrix},
\]  
(21)
and since each entry in (21) is strictly positive, there exist \(c_0 > 0,\) \(c_1 > 0,\) and \(c_2 > 0\) satisfying (5), for all \(2 < k \leq 3.\) For example, when \(k = 2.5\) (\(c = 1.875\)), we have \(h = (4\sqrt{2} - 2, 9\sqrt{3} - 8, 32 - 18\sqrt{3} + 4\sqrt{2}) \approx (3.656854248, 5.27474877, 6.479939708).\) Hence, employing (21) with \(p_2 = 5/72 \approx 0.06944444444\) and \(p_3 = 15/512 \approx 0.02929687500,\) we may take \(c_0 = c_2 = 3.65\) and \(c_1 = 1.82\) in (8).

Example 2. (Exponential rate bounds) Consider \(B = \{B_n\}\) and \(\tilde{B}\) defined by \(B_n = ne^n\) and \(\tilde{B}(x) = xe^x,\) respectively. We then have \(\tilde{B}''(x) = (x+3)e^x,\) and hence, \(\tilde{B}''\) is non-decreasing. Employing Lemma 1, each of the following satisfy the requirements of Theorem 1.

(i) \(p \equiv q \equiv 0\) and for \(n \geq 3,\)
\[
0 \leq r_{n+1} \leq \frac{n + 2}{n - 1}.
\]  
(22)

(ii) \(q \equiv r \equiv 0,\) and for \(n \geq 3,\)
\[
0 \leq p_{n+1} \leq \frac{(n + 2) e^{-2}}{n + 1}.
\]  
(23)

As an example of \(r_n\) satisfying (22), we have \(r_n = \frac{n+1}{n+1}.\) Here, \(h = (2 e^2 - 2 e, 3 e^3 - 4 e^2 + e, 4 e^4 - 6 e^3 + 2 e^2) \approx (9.341548544, 33.41866819, 112.6574908),\) \(r_2 = 3,\) \(r_3 = 2\) and
\[
G = \begin{bmatrix}
1 & 4 & 4 \\
2 & 2 & 0 \\
1 & 1 & 1
\end{bmatrix}.
\]  
(24)
Thus, \(c_0 = 8.35,\) \(c_1 = 4.67,\) and \(c_2 = 9.34\) satisfy (5), and Theorem 1 is applicable.

We now turn to a proof of Theorem 1.

2. Proof of Theorem 1

In this section we will prove Theorem 1.
Prior to proving Theorem 1 we quote the following two tables which we use in the proof of the theorem.

Table 1. Values for \( \{b_i\} \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( b_0 )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
<th>( b_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( c_0 )</td>
<td>0</td>
<td>0</td>
<td>( (1 + r_2)c_0 )</td>
<td>( (3 + p_3)(1 + r_2)c_0 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(-c_1)</td>
<td>0</td>
<td>( (3 + q_2)c_1 )</td>
<td>( (3 + p_3)(3 + q_2)c_1 - (1 + r_3)c_1 )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>( c_2 )</td>
<td>( (3 + p_2)c_2 )</td>
<td>( (3 + p_3)(3 + p_2)c_2 - (3 + q_3)c_2 )</td>
</tr>
</tbody>
</table>

Table 2. Second-order differences for \( \{b_i\} \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( \Delta^2 b_0 )</th>
<th>( \Delta^2 b_1 )</th>
<th>( \Delta^2 b_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( c_0 )</td>
<td>( (1 + r_2)c_0 )</td>
<td>( (1 + p_3)(1 + r_2)c_0 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2c_1 )</td>
<td>( (2 + q_2)c_1 )</td>
<td>( (1 + p_3)(3 + q_2)c_1 - (1 + r_3)c_1 )</td>
</tr>
<tr>
<td>3</td>
<td>( c_2 )</td>
<td>( (1 + p_2)c_2 )</td>
<td>( (1 + p_3)(3 + p_2)c_2 - (2 + q_3)c_2 )</td>
</tr>
</tbody>
</table>

**Proof of Theorem 1.** Suppose \( \{p_i\}, \{q_i\}, \{r_i\}, \{B_i\}, \) and \( (c_0, c_1, c_2) \) satisfy the hypotheses of the theorem. We will consider three cases for \( \{b_i(b_0, b_1, b_2)\} \), namely Case 1: \( \{b_i(c_0, 0, 0)\} \), Case 2: \( \{b_i(0, -c_1, 0)\} \), and Case 3: \( \{b_i(0, 0, c_2)\} \). The values in Tables 1 and 2 follow directly from (2).

Now, note that, for each case, \( b_2 \geq 0, \Delta b_1 \geq 0, \) and by (5), \( \Delta^2 b_i \geq 0, \) for \( i = 0, 1, 2. \) Also, for \( n \geq 2, \) expanding \( b_{n+1} \) via (2), and simplifying, gives

\[
\Delta^2 b_{n-1} = b_{n+1} - 2b_n + b_{n-1}
\]

\[
= \Delta^2 b_{n-2} + \mathcal{L}(b)_{n-1}.
\]

(25)

Assuming that \( \Delta^2 b_i \geq 0 \) for \( i < N - 1, \) gives \( b_i \geq 0 \) for \( 2 \leq i < N + 1 \) and \( \Delta b_i \geq 0 \) for \( 1 \leq i < N. \) Hence, (4) implies that either

\[
\mathcal{L}(b)_{N-1} = p_N b_N - q_N b_{N-1} + r_N b_{N-2}
\]

\[
\geq (p_N - q_N) b_{N-1} + r_N b_{N-2}
\]

\[
\geq 0
\]

(26)

or

\[
\mathcal{L}(b)_{N-1} \geq p_N b_N + (-q_N + r_N) b_{N-2} \geq 0.
\]

(27)

Thus, combining this with the induction hypothesis and (25) gives \( \Delta^2 b_{N-1} \geq 0, \) and the induction is complete. In particular, we have \( \Delta b_i \geq 0 \) for \( i \geq 1 \) and \( b_i \geq 0, \) for \( i \geq 2. \)
Now, for $i \geq 0$, define $\epsilon_i$ by $\epsilon_i \overset{\text{def}}{=} B_i - b_i$. The values of $\epsilon_i$, for the first few $i$, are given in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>$\epsilon_0$</th>
<th>$\epsilon_1$</th>
<th>$\epsilon_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$B_0 - c_0$</td>
<td>$B_1$</td>
<td>$B_2$</td>
</tr>
<tr>
<td>2</td>
<td>$B_0$</td>
<td>$B_1 + c_1$</td>
<td>$B_2$</td>
</tr>
<tr>
<td>3</td>
<td>$B_0$</td>
<td>$B_1$</td>
<td>$B_2 - c_2$</td>
</tr>
</tbody>
</table>

Table 3. Values for $\{\epsilon_i\}$.

We will show that $\epsilon_i \geq 0$ for all $i \geq 3$; the result in (8) then follows, since for general $b_0$, $b_1$, and $b_2$, we then have

$$|b_n(b_0, b_1, b_2)| = \left| \frac{b_0}{c_0} b_n(c_0, 0, 0) - \frac{b_1}{c_1} b_n(0, -c_1, 0) + \frac{b_2}{c_2} b_n(0, 0, c_2) \right| \leq \left| \frac{b_0}{c_0} B_n + \frac{|b_1|}{c_1} B_n + \frac{|b_2|}{c_2} B_n. \right| \tag{28}$$

Note that (5) guarantees that $\Delta^2 \epsilon_i \geq 0$, for $i = 0, 1, 2$ and the assumptions on $B$ give $\Delta \epsilon_0 > 0$ and $\epsilon_1 > 0$ (see Table 3). Now, assume $\Delta^2 \epsilon_n \geq 0$, for $n < N$. It then follows immediately that

$$\epsilon_n \geq \epsilon_{n-1} \geq 0, \tag{29}$$

for $1 \leq n < N + 2$. Hence, we have

$$\Delta^2 \epsilon_N = \Delta^2 B_N - \Delta^2 b_N$$

$$= \Delta^3 B_{N+1} + \Delta^2 B_{N-1} - \Delta^2 b_N$$

$$= \Delta^3 B_{N+1} + \Delta^2 B_{N-1} - b_{N+2} + 2b_{N+1} - b_N$$

$$= \Delta^3 B_{N+1} + \Delta^2 B_{N-1}$$

$$- ((3 + p_{N+1}) b_{N+1} - (3 + q_{N+1}) b_N + (1 + r_N) b_{N-1})$$

$$+ 2b_{N+1} - b_N$$

$$= (\Delta^3 B_{N+1} - p_{N+1} B_{N+1} + q_{N+1} B_N - r_N B_{N-1})$$

$$+ p_{N+1} \epsilon_{N+1} - q_{N+1} \epsilon_N + r_{N+1} \epsilon_{N-1} + (\Delta^2 B_{N-1} - \Delta^2 b_{N-1})$$

$$\geq V(N) + \Delta^2 \epsilon_{N-1} + \Delta^2 \epsilon_{N-1}$$

$$\geq 0. \tag{30}$$

The second to last inequality in (30) follows from (29) and (4). The final inequality follows from (7) and the induction hypothesis. Thus, $\{\epsilon_i\}$ is positive (and convex), and as mentioned, (8) now follows. \qed
Acknowledgements. The first author acknowledges financial support from a Sterge Faculty Fellowship.

References


