Monotone Convex Sequences and Cholesky Decomposition of Symmetric Toeplitz Matrices

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Abstract

This paper studies off-diagonal decay in symmetric Toeplitz matrices. It is shown that if the generating sequence of the matrix is monotone, positive and convex then the monotonicity and positivity are maintained through triangular decomposition. The work is motivated by recent results on explicit bounds for inverses of triangular matrices.

Key words: Monotone Convex Sequences, Cholesky Decomposition, Toeplitz Matrices.


1 Introduction.

Much work has been done in the recent past to understand off-diagonal decay properties of inverses of structured matrices (cf. Benzi and Golub [1], Demko, Moss and Smith [7], Eijkhout and Polman [8], Jaffard [13], Nabben [16] and [17], Peluso and

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Politi [18], Robinson and Wathen [21], Strohmer [23], Vecchio [27] and the references therein). Recent results bounding entries in inverses of triangular matrices given bounds on ratios of off-diagonal to diagonal entries in the original matrix (cf. Berenhaut and Lund [3], Berenhaut and Morton [4] and Berenhaut and Fan [5]), suggest investigation of instances when decay is preserved under triangular decomposition. In this paper we take some steps in the direction, by considering symmetric Toeplitz matrices. In particular, we will prove the following theorem.

**Theorem 1.** Consider an $n \times n$ symmetric Toeplitz matrix $T = [t_{i,j}]$ generated by a sequence $\{X_i\}_{i=0}^n$, i.e. $t_{i,j} = X_{|i-j|}$ for $1 \leq i, j \leq n$:

$$T = \begin{bmatrix}
X_0 & X_1 & X_2 & \cdots & X_n \\
X_0 & X_1 & \cdots & X_{n-1} \\
& \ddots & \ddots & \ddots \\
X_0 & X_1 & & \\
\text{symm.} & & & X_0
\end{bmatrix}. \tag{1}$$

Suppose the sequence $\{X_0, X_1, \cdots, X_n\}$ satisfies the relations:

(a) **Monotonicity and positivity:**

$$X_0 \geq X_1 \geq X_2 \geq \cdots \geq X_n \geq 0 \tag{2}$$

and

(b) **Convexity:**

$$X_0 - X_1 \geq X_1 - X_2 \geq \cdots \geq X_{n-1} - X_n \geq 0 \tag{3}$$

then the Cholesky decomposition of the matrix $T$, given by

$$T = LL', \tag{4}$$
where $L = [l_{i,j}]$ is lower-triangular, satisfies

$$l_{i,j} \geq 0, \quad (1 \leq j \leq i \leq n)$$

and

$$l_{i,j} \geq l_{i+1,j}, \quad (1 \leq j \leq i \leq n - 1).$$

That is, the monotonicity and positivity off the diagonal are maintained through the decomposition.

Note that via the Carathéodory-Toeplitz Theorem (cf. Grenander and Szego [12]), (3) implies that $T$ is positive definite (cf. the argument in Lopez-Marcos [14]), and hence the Cholesky decomposition in (4) exists (cf. Golub and Van Loan [9]).

As a by-product of the proof (in particular, of formulas (23) and (42)), we have the following lower-bounds.

**Corollary 1** Under the assumptions in Theorem 1,

$$l_{jj}l_{kj} \geq (X_{k-j} - X_{k-(j-1)}) + X_{k-1}(X_{j-2} - X_{j-1}),$$

for $k \geq j$.

In particular, setting $k = j$ in (7), we have a lower bound on the diagonal entries in $L$,

$$l_{jj} \geq \sqrt{(X_0 - X_1) + X_{j-1}(X_{j-2} - X_{j-1})},$$

The following example gives some computational evidence that convexity of the sequence $\{X_i\}$ is indeed important in ensuring that the monotonicity and positivity are inherited after triangular decomposition.
Table 1
Results of simulations for Example 1

<table>
<thead>
<tr>
<th>n</th>
<th>repetitions</th>
<th>permutations</th>
<th>convex</th>
<th>positive definite matrices</th>
<th>positive monotone and $(5) &amp; (6)$</th>
<th>proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1000</td>
<td>120</td>
<td>5</td>
<td>72.88(23.18)</td>
<td>16.25(10.28)</td>
<td>.222(.096)</td>
</tr>
<tr>
<td>6</td>
<td>200</td>
<td>720</td>
<td>6</td>
<td>387.14(123.04)</td>
<td>31.12(22.05)</td>
<td>.080(.052)</td>
</tr>
<tr>
<td>7</td>
<td>100</td>
<td>5040</td>
<td>7</td>
<td>2181.63(817.09)</td>
<td>71.40(70.51)</td>
<td>.032(.024)</td>
</tr>
<tr>
<td>8</td>
<td>50</td>
<td>40320</td>
<td>8</td>
<td>15055.63(6679.70)</td>
<td>106.50(118.50)</td>
<td>.0063(.0037)</td>
</tr>
</tbody>
</table>

**Example 1.** Consider $n$ distinct non-negative real numbers $y_0, y_1, \cdots, y_{n-1}$. For a re-ordering $z_0, z_1, \cdots, z_{n-1}$ of $y_0, y_1, \cdots, y_{n-1}$, the sequence $X_0, X_1, \cdots, X_{n-1}$ where

$$X_i = \sum_{j=0}^{n-i-1} z_i,$$

for $0 \leq i \leq n - 1$, satisfies (2). Of the $n!$ different sequences $\{X_j\}$ which result from re-orderings of $\{y_j\}$, $n$ are convex (one for each of the $n$ possible values of $z_0$).

To investigate the near necessity of the convexity assumption (3) in the statement of Theorem 1, we generated several random samples $y_0, y_1, \cdots, y_{n-1}$ of random variables distributed uniformly on the interval $(0,1)$ (for $n = 5, 6, 7$ and 8). All positive definite matrices $T$ resulting from (1) were checked for (5) and (6). The results are summarized in Table 1. The last three columns contain averages over the associated repetitions; the numbers in parentheses are the respective sample standard deviations. Note that while all $n$ convex permutations lead to Toeplitz matrices with Cholesky decompositions satisfying (5) and (6), the proportions were quite small overall.

$\square$

**Remark.** Note that Cholesky factors of Toeplitz matrices have been studied by several authors in the recent past (cf. Bojańczyk et al. [6], Goodman et al. [10], [11], Koltracht and Neumann [15], Stewart [22] and van der Mee et al. [24]–[26]). When
off-diagonal decay properties are discussed, the results are often asymptotic in nature and arise from analytic considerations. It appears that these techniques are not easily amenable to a study of the local conditions of interest here, namely monotonicity and convexity.

The remainder of the paper proceeds as follows. Section 2 contains some preliminary lemmas, while Section 3 consists of the proof of Theorem 1. The proof makes use of a recent result of the authors on bounds for recurrences with monotone coefficients.

2 Preliminary Lemmas

From here onwards, we will assume \( \{X_i\}_{i=0}^n \) satisfies (2) and (3). As well, it is sufficient to assume, without loss of generality, that \( X_0 = 1 \).

We will need the following straightforward lemma.

**Lemma 1** For \( 0 \leq j < k \leq n \), set

\[
\Delta_{k,j} \overset{\text{def}}{=} X_{k-1+j} - X_{k-1}X_j
\]  

(10)

and

\[
\Delta^*_{k,j} \overset{\text{def}}{=} \Delta_{k,j} - \Delta_{k+1,j} = (X_{k-1+j} - X_{k-j}) - X_j(X_{k-1} - X_k).
\]  

(11)

Then, for \( 1 \leq k \leq n \),

\[
0 = \Delta_{k,0} \leq \Delta_{k,1} \leq \cdots \leq \Delta_{k,k-1}
\]  

(12)

and

\[
0 = \Delta^*_{k,0} \leq \Delta^*_{k,1} \leq \cdots \leq \Delta^*_{k,k-1}.
\]  

(13)
Proof. We will prove (13); the proof of (12) is similar. For $0 \leq j \leq k - 2$, we have

\[
\Delta^*_{k,j+1} - \Delta^*_{k,j} = [(X_{k-(2+j)} - X_{k-(1+j)}) - (X_{k-1} - X_k)X_{j+1}] \\
- [(X_{k-(1+j)} - X_{k-j}) - (X_{k-1} - X_k)X_j] \\
= [(X_{k-(2+j)} - X_{k-(1+j)}) - (X_{k-(1+j)} - X_{k-j})] \\
+ (X_{k-1} - X_k)(X_j - X_{j+1}) \\
\geq 0.
\] (14)

The inequality in (14) follows by the convexity and monotonicity assumptions in (3) and (2). Also, since $X_0 = 1$,

\[
\Delta^*_{k,0} = (X_{k-1} - X_k)(1 - X_0) = 0.
\] (15)

\[\square\]

The following technical lemma on recursively defined functions is crucial.

**Lemma 2** Suppose $\{\Lambda_i\}_{i \geq 1}$ and $\{\alpha_{i,j}\}_{2 \leq j \leq i-1}$, are constants, and recursively define the functions $\{S_i\}_{i > 1}$ by $S_2(\Lambda_1) = \Lambda_1$, and for $i > 2$

\[
S_i(\Lambda_1, \Lambda_2, \cdots, \Lambda_{i-1}) = \Lambda_{i-1} + \sum_{j=2}^{i-1} \alpha_{i,j}S_j(\Lambda_1, \Lambda_2, \cdots, \Lambda_{j-1}).
\] (16)

Then,

\[
S_i(\Lambda_1, \Lambda_2, \cdots, \Lambda_{i-1}) = \sum_{j=1}^{i-1} d_{i,j}\Lambda_j,
\] (17)

for $i > 1$, where $d_{i,i-1} = 1$ and for $1 \leq j \leq i - 2$,

\[
d_{i,j} = \sum_{v=j+1}^{i-1} \alpha_{v+1,j+1}d_{i,v}.
\] (18)
Note that in (18) \( d_{i,j} \) is expressed as a linear combination of \( d_{i,j+1}, \ldots, d_{i,i-1} \), where \( d_{i,i-1} = 1 \) is the initial value for the recurrence.

**Proof of Lemma 2.** First, setting \( c_{i,i-1} = 1 \) and

\[
c_{i,j} = \sum_{v=j+1}^{i-1} \alpha_{i,v} c_{v,j}.
\]

(19)

for \( 1 \leq j \leq i - 2 \), we have from (16) that

\[
S_i(\Lambda_1, \Lambda_2, \ldots, \Lambda_{i-1}) = \sum_{j=1}^{i-1} c_{i,j} \Lambda_j,
\]

(20)

is satisfied for all \( i > 1 \).

Now, direct computation with (16) gives that (17) is true for \( i = 2 \) and \( i = 3 \). Hence, suppose that \( N \geq 4 \) and (17) is true for \( 1 \leq j < i \leq N - 1 \).

Recall that \( c_{i,i-1} = d_{i,i-1} = 1 \) for all \( i \), and in particular, \( d_{N,N-1} = c_{N,N-1} = 1 \). Hence, assume \( 1 \leq J \leq N - 2 \) and

\[
d_{N,j} = c_{N,j}
\]

(21)

holds for \( J + 1 \leq j \leq N - 1 \). We will show that (21) holds for \( j = J \).

Employing the induction hypothesis and (19) and swapping summation gives
\[ d_{N,J} = \sum_{v=J+1}^{N-1} \alpha_{v+1,J+1} d_{N,v} \]
\[ = \sum_{v=J+1}^{N-1} \alpha_{v+1,J+1} c_{N,v} \]
\[ = \sum_{v=J+1}^{N-2} \alpha_{v+1,J+1} \sum_{w=v+1}^{N-1} \alpha_{N,w} c_{w,v} + \alpha_{N,J+1} c_{N,J+1} \]
\[ = \sum_{w=J+2}^{N-1} \alpha_{N,w} \sum_{v=J+1}^{w-1} \alpha_{v+1,J+1} c_{w,v} + \alpha_{N,J+1} \]
\[ = \sum_{w=J+2}^{N-1} \alpha_{N,w} d_{w,J} + \alpha_{N,J+1} \]
\[ = \sum_{w=J+1}^{N-1} \alpha_{N,w} d_{w,J} \]
\[ = \sum_{w=J+1}^{N-1} \alpha_{N,w} c_{w,J} \]
\[ = c_{N,J}. \]  

(22)

Since \( 1 \leq J \leq N - 1 \) was arbitrary, the lemma now follows. \( \square \)

3 Proof of The Main Result

This section contains the proof of Theorem 1

**Proof of Theorem 1.** For \( 1 \leq j \leq i \leq n \), define

\[ R_{i,j} \overset{\text{def}}{=} l_{i,j} l_{i,j}. \]  

(23)

It follows from standard Cholesky decomposition formulas (cf. Golub and van Loan [9] or Press et al. [20]) and (11) that
\[ R_{i,j} = X_{i-j} - \sum_{v=1}^{j-1} l_{i,v} l_{j,v} \]
\[ = X_{i-j} - l_{i,1} l_{j,1} - \sum_{v=2}^{j-1} l_{i,v} l_{j,v} \]
\[ = X_{i-j} - X_{i-1} X_{j-1} - \sum_{v=2}^{j-1} l_{j,v} l_{i,v} \]
\[ = \Delta_{i,j-1} - \sum_{v=2}^{j-1} l_{j,v} R_{i,v} \] \hspace{1cm} (24)

Setting
\[ \beta_{j,v} \stackrel{\text{def}}{=} -\frac{l_{j,v}}{l_{v,v}} \] \hspace{1cm} (25)
we have
\[ R_{i,j} = \Delta_{i,j-1} + \sum_{v=2}^{j-1} \beta_{j,v} R_{i,v} \] \hspace{1cm} (26)

Since \( T \) is assumed to be positive definite, we have \( l_{j,j} > 0 \) for all \( 1 \leq j \leq n \) (cf. Golub and Van Loan [9]). Hence, (5) and (6) are equivalent to

\[ R_{i,j} \geq 0, \quad (1 \leq j \leq i \leq n) \] \hspace{1cm} (27)

and

\[ R_{i,j} \geq R_{i+1,j}, \quad (1 \leq j \leq i \leq n-1), \] \hspace{1cm} (28)

respectively.

We shall prove (27) and (28). Note that \( R_{i,1} = X_0 X_{i-1} \geq 0 \). Now, fix \( K \) such that \( 2 \leq K \leq n \). Note that by (24) and Lemma 1,

\[ R_{K,2} = \Delta_{K,1} \geq 0 \] \hspace{1cm} (29)
and if $2 \leq K < n$,

$$
R_{K,2} - R_{K+1,2} = \Delta_{K,1} - \Delta_{K+1,1} \\
= (X_{K-2} - X_{K-1}X_1) - (X_{K-1} - X_KX_1) \\
= (X_{K-2} - X_{K-1}) - X_1(X_{K-1} - X_K) \\
= \Delta^*_K,1 \\
\geq 0.
$$  \hspace{1cm} (30)

Now, assume that (27) and (28) are true for $i = K$ and $j < J$. Applying Lemma 2 to (26) with $\Lambda_v = \Delta_{K,v}$ and $\alpha_{v,k} = \beta_{v,k}$ for $2 \leq k \leq v - 1$, gives

$$
R_{K,j} = \sum_{v=1}^{j-1} d_{j,v} \Delta_{K,v}, 
$$  \hspace{1cm} (31)

for $2 \leq j \leq K$, where $d_{j,j-1} = 1$ and for $1 \leq v \leq j - 2$,

$$
d_{j,v} = \sum_{w=v+1}^{j-1} \beta_{w+1,v+1} d_{j,w}.  \hspace{1cm} (32)
$$

and, upon taking differences in (31),

$$
R_{K,j} - R_{K+1,j} = (\Delta_{K,j-1} - \Delta_{K+1,j-1}) + \sum_{v=1}^{j-2} d_{j,v}(\Delta_{K,v} - \Delta_{K+1,v}) \\
= \sum_{v=1}^{j-1} d_{j,v}\Delta^*_K,v.  \hspace{1cm} (33)
$$

Note that by the induction hypotheses, $\beta_{i,j} \in [-1, 0]$ for all $j \leq i < J$, and
\[ \beta_{i,j} = -\frac{R_{i,j}}{R_{j,j}} \geq -\frac{R_{i-1,j}}{R_{j,j}} = \beta_{i-1,j} \] (34)

for \( j \leq i < J - 1. \)

Summarizing (31), (32) and (34), we have

\[
R_{K,j} = \sum_{v=1}^{j-1} d_{j,v} \Delta_{K,v} = \sum_{v=1}^{j-1} d_{j,v} \Delta_{K,j-v} = \Delta_{K,j-1} + \sum_{v=2}^{j-1} d_{j,v} \Delta_{K,j-v} = \Delta_{K,j-1} - \left[ \sum_{v=2}^{j-1} (-d_{j,v}) \Delta_{K,j-v} \right] \] (35)

where \(-d_{j,j-1} = -1\) and

\[-d_{j,j-v} = \sum_{p=1}^{v-1} \beta_{j-p+1,j-v+1}(-d_{j,v}) \] (36)

for \( 2 \leq v \leq j - 1, \) and

\[ 0 \geq \beta_{j,v+1} \geq \beta_{j-1,v+1} \geq \cdots \geq \beta_{j-v+2,j-v+1} \geq -1 \] (37)

We will use the following inequality for recurrences with monotone coefficients which is a slight restatement of that in Berenhaut et al. [2].

**Theorem 2** Suppose \( \{x_i\} \) is a non-increasing, non-negative sequence, \( \{b_i\} \) satisfies \( b_1 = -1 \) and
\[ b_v = \sum_{p=1}^{v-1} \gamma_{v,p} b_p, \quad (v \geq 2) \]  

(38)

with

\[ 0 \geq \gamma_{v,1} \geq \gamma_{v,2} \geq \cdots \geq \gamma_{v,v-1} \geq -1 \]  

(39)

for \( v \geq 2 \), then

\[ \sum_{v=2}^{N} b_v x_v \leq x_2. \]  

(40)

**Proof.** See Berenhaut *et al.* [2]. \( \square \)

Now, we may apply Theorem 2 with

\[
\begin{align*}
 b_v &= -d_{j,j-v} \\
x_v &= \Delta_{K,j-v} \\
\gamma_{v,p} &= \beta_{j-p+1,j-v+1}
\end{align*}
\]  

(41)

to obtain

\[
\begin{align*}
 R_{k,j} &= \Delta_{K,j-1} - \left[ (-d_{j,j-2})\Delta_{K,j-2} + \cdots + (-d_{j,1})\Delta_{K,1} \right] \\
&\geq \Delta_{K,j-1} - \Delta_{K,j-2} \\
&\geq 0.
\end{align*}
\]  

(42)

Similar arguments with (33) in place of (31) show that

\[
\begin{align*}
 R_{k,j} - R_{k+1,j} &= \Delta^*_{K,j-1} - \left[ (-d_{j,j-2})\Delta^*_{K,j-2} + \cdots + (-d_{j,1})\Delta^*_{K,1} \right] \\
&\geq \Delta^*_{K,j-1} - \Delta^*_{K,j-2} \\
&\geq 0.
\end{align*}
\]  

(43)
This completes the proof of Theorem 1.

\[ \blacksquare \]

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References


