Second Order Bounds for Linear Recurrences with Negative Coefficients

Kenneth S. Berenhaut a,*, Daniel C. Morton a

aWake Forest University, Department of Mathematics, Winston-Salem, NC 27109

Abstract

This paper introduces a generalization of Fibonacci and Pell polynomials in order to obtain optimal second-order bounds for general linear recurrences with negative coefficients. An important aspect of the derived bounds is that they are applicable and easily computable. The results imply bounds on all entries in inverses of triangular matrices as well as on coefficients of reciprocals of power series.

Key words: Recurrence, Restricted Coefficients, Negative Coefficients, Power Series, Triangular Matrices, Fibonacci Polynomials.


1 Introduction.

This paper studies general linear recurrences of the form

\[ b_n = \sum_{k=1}^{n-1} \alpha_{n,k} b_k, \quad (n \geq 2) \]

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* Corresponding author.

Email addresses: berenhks@wfu.edu (Kenneth S. Berenhaut), mortdc0@wfu.edu (Daniel C. Morton).

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where for some $A > 0$,

$$\alpha_{n,k} \in [-A, 0] \text{ for } 1 \leq k \leq n - 1; n \geq 2.$$  \hspace{1cm} (2)

Without loss of generality we will assume that $b_1 = -1$. Here we will show that solutions to (1) and (2) must, by necessity, eventually be bounded by a simple second order linear recurrence. More specifically, we will prove the following theorem.

**Theorem 1** Suppose that $A > 0$ and $m = \lfloor 1/A \rfloor$. If $\{U_j\}_{j=1}^\infty$ is defined by

$$U_n = \max\{|b_n| : \{b_i\} \text{ and } \{\alpha_{i,j}\} \text{ satisfy (1) and (2)}\}; n \geq 2,$$

then

$$U_n = \begin{cases} A, & \text{if } n = 2 \\ \max(A, A^2), & \text{if } n = 3 \\ \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor A^3 + A, & \text{if } 4 \leq n \leq 2m + 2 \\ (n - 2)A^2, & \text{if } n = 2m + 3 \\ AU_{n-1} + U_{n-2}, & \text{if } n \geq 2m + 4 \end{cases}$$

\hspace{1cm} (4)

□

Note that the point at which the second order recurrence bound in (4) takes effect depends on the value of $m$ (and hence on $A$). Still, optimal bounding of (1) has been reduced to solving a simple second order homogeneous linear recurrence.

\textsuperscript{1} Square brackets indicate the greatest integer function.
The general solution to the linear recurrence

\[ V_n = AV_{n-1} + V_{n-2}, \quad n \geq 2 \]  

with starting values \( V_0 \) and \( V_1 \) is

\[
V_n = \delta^{-1} \left( V_1 - AV_0 + 2 \frac{V_0}{\delta - A} \right) \left( \frac{2}{\delta - A} \right)^n \\
+ \delta^{-1} \left( AV_0 - V_1 + 2 \frac{V_0}{\delta + A} \right) \left( \frac{-2}{\delta + A} \right)^n,
\]

where \( \delta = \sqrt{A^2 + 4} \).

The proof of Theorem 1 essentially involves reducing the problem to considering only \( \alpha_{i,j} \) on the boundary of \([-A, 0]\) (ie. \( \alpha_{i,j} \in \{-A, 0\} \), for \( 1 \leq j \leq i - 1; \ i \geq 2 \)) and then comparing the multitude of polynomials in \( A \) (see Table 2 below) which may result from (1). As in Berenhaut and Lund [3] we will rely heavily on sign change analyses.

Recurrences with varying or random coefficients have been studied by many previous authors. A partial survey of such literature contains Viswanath [27] and [28], Viswanath and Trefethen [29], Embree and Trefethen [7], Wright and Trefethen [31], Mallik [20], Popenda [24], Kittapa [19], and Odlyzko [23].

While of interest from a theoretical standpoint, bounds for recurrences such as those in (1) can be useful in a range of applications through connections to triangular matrix equations and (multiplicative) inversion of power series. These connections are discussed in Section 2. The final three sections comprise a proof of Theorem 1.
2 Applications of Theorem 1 to power series and matrix inversion

2.1 Reciprocals of power series

For a fixed \( I \subset \mathbb{R} \), let \( \mathcal{F}_I \) be the set of \( I \)-power series defined by

\[
\mathcal{F}_I = \{ f : f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \text{ and } a_k \in I \text{ for each } k \geq 1 \}. \quad (7)
\]

Flatto, Lagarias, and Poonen [9] and Solomyak [26] proved independently that if \( z \) is a root of a series in \( \mathcal{F}_{[0,1]} \), then \( |z| \geq 2/(1 + \sqrt{5}) \). As \( z = -2/(1 + \sqrt{5}) \) is a root of \( 1 + z + z^3 + z^5 + \cdots \), this bound is tight over \( \mathcal{F}_{[0,1]} \). The coefficients of the multiplicative inverse of a series in \( \mathcal{F}_{[0,1]} \) cannot increase at a rate larger than the golden ratio.

Now, consider computation of the coefficients of the multiplicative inverse \( h \) of a series \( f \in \mathcal{F}_{[0,1]} \). Equating coefficients in the expansion

\[
h_0 + h_1 z + h_2 z^2 + \cdots = \frac{1}{1 + f_1 z + f_2 z^2 + \cdots} \quad (8)
\]

gives \( h_0 = 1 \) and

\[
h_n = -\sum_{j=0}^{n-1} f_{n-j} h_j, \quad n \geq 1
\]

\[
= \sum_{j=0}^{n-1} (-f_{n-j}) h_j. \quad (10)
\]

Theorem 1 is then directly applicable and we obtain
Corollary 1 Suppose that $A > 0$, $m = [1/A]$, $f \in \mathcal{F}_{[0,A]}$ and $f$ and $h$ satisfy (8). Then,

$$|h_n| \leq U_{n+1}$$

(11)

for all $n \geq 0$ where $\{U_n\}_{n=1}^{\infty}$ is as in (4).

The rate of growth of the bound in (11) is optimal (see Corollary 2). In fact, the bound is actually best possible when $A \geq 1$ (see Berenhaut and Lund [3]). It is not difficult to show that, in general, if $f$ and $h$ satisfy (8) with

$$f(z) = 1 + Az + Az^3 + Az^5 + \cdots,$$

(12)

then

$$1 \leq \frac{U_{n+1}}{|h_n|} \leq \max(1, \frac{1}{A}).$$

(13)

Figure 1 gives a plot of the ratios in (13) for $A = 0.1$ and $0 \leq n \leq 100.$
Corollary 1 may be useful where generating functions or formal power series are utilized such as in enumerative combinatorics and stochastic processes (cf. Wilf [30], Feller [8], Kijima [18], Heathcote [14], Kendall [17]).

The above results provide bounds for the location of the smallest root of a complex valued power series. Power series with restricted coefficients have been studied in the context of determining distributions of zeroes (cf. Flatto et al. [9], Solomyak [26], Beaucoup et al. [1], [2], and Pinner [25]). Related problems for polynomials have been considered by Odlyzko and Poonen [22], Yamamoto [32], Borwein and Pinner [6], and Borwein and Erdelyi [5]. As mentioned above, Flatto et al. [9] and Solomyak [26] independently proved that if \( z \) is a root of a series in \( \mathcal{F}_{[0,1]} \), then \( |z| \geq 2/(1 + \sqrt{5}) \). The following extension of this result is a consequence of Corollary 1.

**Corollary 2** Let \( C = 2/(\delta + A) \), where \( \delta = \sqrt{A^2 + 4} \). If \( z \) is a root of a power series in \( \mathcal{F}_{[0,A]} \) with \( 0 \leq A \), then

\[
|z| \geq C.
\]  

(14)

The result in Corollary 2 is optimal: for given \( 0 \leq A \), \( f(z) = 1 + Az + Az^3 + Az^5 + \cdots \) has a root at \( z = -C \).

**Proof of Corollary 2.** Suppose that \( f \in \mathcal{F}_{[0,A]} \). Apply Corollary 1, and note from (6) that \( f(z)^{-1} \) is finite for \( |z| < C \). If \( f \) had a root in \( \{ z : |z| < C \} \), say at \( z = z_0 \), then we would have the contradiction \( |f(z_0)|^{-1} = \infty \).

**Remark.** Note that Corollary 2 extends a result from Berenhaut and Lund [3] where the simpler case \( A \geq 1 \) was proved.
2.2 Bounds for entries of inverses of triangular matrices

Theorem 1 also has applications to bounding inverses of triangular matrices. These will be discussed in [4], but we include the following result as an example.

**Corollary 3** Consider inverting the lower triangular matrix $L_n = [l_{i,j}]_{n \times n}$; ie. solving for $X_n = [x_{i,j}]_{n \times n}$ in the lower triangular linear system $L_n X_n = I_n$, where $I_n$ is the $n \times n$ identity matrix. If $l_{i,j} / l_{i,i} \in [0, A]$ for $1 \leq i \leq n$ and $1 \leq j \leq i - 1$, then

$$|x_{k,s}| \leq \frac{U_{k-s+1}}{l_{s,s}}$$

(15)

for $1 \leq s \leq n$ and $s \leq k \leq n$, where $\{U_i\}$ is as in (4).

Corollary 3 compares favorably to bounds for matrix equation solutions with entries that are restricted to more general intervals in Neumaier [21], Hansen [11] and [10], Hansen and Smith [12], and Kearfott [16]. Here, optimal bounds are obtained regardless of interval widths and dimension; moreover, the computational burden is limited to solving the second-order linear recurrences in (4).

3 Preliminaries for a proof of Theorem 1

Suppose that $\{b_i\}$, $\{\alpha_{i,j}\}$ and $A > 0$ satisfy (1) and (2) with $b_1 = -1$.

Let $\mathcal{P} = \{n \geq 1 : b_n \geq 0\}$ and $\mathcal{N} = \{n \geq 1 : b_n < 0\}$ partition the sign configuration of $\{b_i\}_{i=1}^{\infty}$. Now, define $B_n$ (a polynomial in $A$) recursively in $n$ from $\mathcal{N}$ and $\mathcal{P}$ via $B_1 = -1$ and
\[ B_n = \begin{cases} 
A + A \sum_{2 \leq r \leq n-1} (-B_r), & n \in \mathcal{P} \\
-A \sum_{2 \leq r \leq n-1} B_r, & n \in \mathcal{N} 
\end{cases} \quad (16) \]

for \( n \geq 2 \). A simple induction with (16) will show that \( B_n \) and \( b_n \) have the same sign for \( n \geq 1 \).

The following lemma (similar to Corollary 2.3 in Berenhaut and Lund [3]) reduces the problem of bounding \(|b_n|\) to a comparison of the at most \( 2^{n-1} \) possible distinct “sign configurations” of \( b_2, b_3, \cdots, b_n \).

**Lemma 1** We have

\[ |b_i| \leq |B_i| \quad (17) \]

for all \( i \geq 1 \).

**Proof.** First, note that under the inherent assumptions, \( b_1 = -1 = B_1 \) and \( b_2 = -\alpha_{2,1} \leq A = B_2 \). We shall prove the lemma by induction. Suppose that \( n > 1 \) and that (17) is satisfied for all \( i \leq n - 1 \). Now, assume that \( n \in \mathcal{P} \). Returning to (1) and collecting positive and negative terms gives

\[ b_n = \alpha_{n,1}b_1 + \sum_{2 \leq r \leq n-1} \alpha_{n,r}b_r + \sum_{2 \leq r \leq n-1} \alpha_{n,r}b_r. \quad (18) \]

Using \( b_1 = -1 \), the bound \( \alpha_{n,k} \in [-A, 0] \) for all \( n, k \), and neglecting the first summation in (18) gives
Table 1
Polynomials for \( n = 3, 4, 5 \) and \( 6 \)

\[
\begin{array}{l}
\text{n = 3}: \ 0, A, A^2 \\
\text{n = 4}: \ 0, A, A^3 + A, 2A^2, A^2 \\
\text{n = 5}: \ 0, A, A^3 + A, 2A^2, A^2, 2A^3 + A, A^4 + 2A^2, 3A^2 \\
\text{n = 6}: \ 0, A, A^3 + A, 2A^2, A^2, 2A^3 + A, A^4 + 2A^2, 3A^2, 3A^3 + A, A^5 + 3A^3 + A, 4A^3 + A, 4A^2, 4A^4 + 3A^2, 2A^4 + 2A^2.
\end{array}
\]

\[
b_n \leq A + \sum_{2 \leq r \leq n-1} (-A)b_r
\]

\[
= A + A \sum_{2 \leq r \leq n-1} |b_r|.
\]  \hspace{1cm} (19)

Using the inductive hypothesis and the fact that \( |b_n| = b_n \) in (19) produces

\[
|b_n| \leq A + A \sum_{2 \leq r \leq n-1} |B_r|
\]

\[
= A - A \sum_{2 \leq r \leq n-1} B_r
\]

\[
= B_n
\]  \hspace{1cm} (20)

after (16) is applied. An analogous argument works when \( n \in \mathbb{N} \). \( \square \)

As mentioned earlier, Lemma 1 reduces our work to searching through the at most \( 2^{n-1} \) polynomials corresponding to the various possible sign configurations of \( b_2, b_3, \ldots, b_n \) to locate the one of largest modulus. Table 1 gives the various polynomials under consideration for \( n = 3, 4, 5 \) and \( 6 \). The number of distinct polynomials for the first few \( n \geq 3 \) are \( 3, 5, 8, 14, 23, 43, 77, \ldots \). Even for a fixed value of \( A \), and \( n \) as low as say 15 the task of computationally searching for the polynomial with the largest value could be quite
Table 2
Coefficients of polynomials for $n = 9$

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The coefficients (on increasing powers of $A$) of the 77 polynomials for $n = 9$ are listed in Table 2.

A quick glance at the list in Table 2 suggests elimination of several polynomials, and the
Table 3

Coefficients of eight selected polynomials for $n = 9$

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Fig. 2. Polynomials in Table 3 near $A = 0.50$.

list can quite quickly be brought down to a manageable size for consideration (see Table 3). Note that for sufficiently large values of $A$ the dominant polynomial will be the one of highest degree (ie. $A^8 + 6A^6 + 10A^4 + 4A^2$). The results of Berenhaut and Lund [3] imply that this polynomial (which corresponds to an alternating sign configuration: $\mathcal{N} = \{1, 3, 5, 7, 9\}$, $\mathcal{P} = \{2, 4, 6, 8\}$) will dominate for all $A \geq 1$. The situation is much more complicated for $A < 1$. Figure 2 shows several crossings in a small interval near $A = 0.5$. Table 4 gives the maximal polynomials with their respective ranges of dominance.
Table 4
Maximal Polynomials for $n = 9$

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<th>Polynomial</th>
<th>Range of dominance</th>
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<tbody>
<tr>
<td>$A^8 + 6A^6 + 10A^4 + 4A^2$</td>
<td>$A \in [1, \infty]$</td>
</tr>
<tr>
<td>$4A^6 + 12A^4 + 5A^2$</td>
<td>$A \in [1/2, 1]$</td>
</tr>
<tr>
<td>$9A^4 + 6A^2$</td>
<td>$A \in [1/3, 1/2]$</td>
</tr>
<tr>
<td>$7A^2$</td>
<td>$A \in [1/4, 1/3]$</td>
</tr>
<tr>
<td>$12A^3 + A$</td>
<td>$A \in (0, 1/4]$</td>
</tr>
</tbody>
</table>

Figure 3 illustrates the information in Table 4. Notice that, while the differences between these polynomials have roots at various values of $A \in (0, \infty]$ the dominant polynomial only changes as $A$ passes through the reciprocal of a whole number. This rather intriguing characteristic continues for all values of $n$.

In order to locate the largest polynomial, we will introduce the following notation. For each partition (or two-colouring) of the set $\{1, 2, 3, \ldots, n\}$ into two sets $N$ and $P$ with
1 ∈ \mathcal{N}$, let \( \{a_i\}_{i \geq 0} \) denote the “string lengths” of the partition. For example, suppose $n = 9$, $\mathcal{N} = \{1, 2, 5, 6, 7\}$ and $\mathcal{P} = \{3, 4, 8, 9\}$. The sequence of string lengths would then be \((a_0, a_1, a_2, a_3) = (2, 2, 3, 2)\) since $1, 2 \in \mathcal{N}$, $3, 4 \in \mathcal{P}$, $5, 6, 7 \in \mathcal{N}$, and $8, 9 \in \mathcal{P}$. Thought of as conveying the sign structure of \( \{b_n\} \) these particular \( \mathcal{N} \) and \( \mathcal{P} \) would correspond to $(-, -, +, +, -, -, -, +, +)$.

According to (16), the sign structure in the preceding paragraph gives

\[
(B_1, B_2, \cdots, B_9) = (-1, 0, A, A, -2A^2, -2A^2, -2A^2, 6A^3 + A, 6A^3 + A).
\] (21)

The value of $6A^3 + A$ for $B_9$ can be found in Table 2, fifth from the top on the right.

Given sets \( \mathcal{N} \) and \( \mathcal{P} \), some properties of the sequence \( \{B_i\} \) follow directly from the definition in (16). First, all values of $B_i$, for $i$ within a string will always be the same. Also, we may assume that $B_n$ and $B_{n-1}$ are of opposite sign (i.e. the final $a_i$ is one), since otherwise flipping the sign of $B_{n-1}$ can not decrease the value of $|B_n|$. We may also assume that $B_1$ and $B_2$ are of opposite sign (i.e. $a_0 = 1$) since otherwise $B_2 = 0$ and flipping its sign will lead to $B_2 = A$ which can only have a positive effect on $|B_i|$, $i \geq 3$.

Finally, let $S_0 = 1$ and $S_j = 1 + a_1 + \cdots + a_j$ for $j \geq 1$. Then, for $1 < k \leq S_1$, $|B_k| = A$ while for $j \geq 1$ and $S_j < k \leq S_{j+1}$

\[
B_k = -Aa_jB_{S_j-1+1} + B_{S_{j-2}+1}.
\] (22)

In fact, we have the following characterization of the values of $|B_n|$, $n \geq 2$.

**Lemma 2** Define the polynomial valued functions $\phi_1(a_1) = a_1A^2$, $\phi_2(a_1, a_2) = a_1a_2A^3 + A$ and for $3 \leq j \leq k$
\[ \phi_j(a_1, a_2, \cdots, a_j) = A a_j \phi_{j-1}(a_1, a_2, \cdots, a_{j-1}) + \phi_{j-2}(a_1, a_2, \cdots, a_{j-2}). \]  

(23)

The non-zero polynomials in \( A \) that may arise as values for \( |B_n| \) for \( n \geq 2 \) are precisely the first degree polynomial \( A \) along with the possible values of \( \phi_k(a_1, a_2, \cdots, a_k) \) for \( k \in \{1, 2, \cdots, n-2\} \) and \( a_1, \cdots, a_k \geq 1 \) such that \( \sum_{v=1}^{k} a_v = n-2 \). \( \Box \)

**Remark 1** For a sequence \( \{a_i\} \), define the sequence of polynomials \( \{\tilde{\phi}_j(a_1, a_2, \cdots, a_j)\} \) by

\[ \tilde{\phi}_j(a_1, a_2, \cdots, a_j) = \phi_j(a_1, a_2, \cdots, a_j) \frac{A}{A}. \]  

(24)

The polynomials in (24) include the Fibonacci \( (a_i \equiv 1) \) and Pell \( (a_i \equiv 2) \) polynomials (cf. Horadam and Mahon [15]) as specific examples. In fact Propositions 1 and 2, below, imply that for fixed \( n \) and \( A \geq 1 \), the Fibonacci polynomial \( \tilde{\phi}_k(1, 1, \cdots, 1) \), dominates all other polynomials satisfying (24) with \( \sum_{v=1}^{i} a_v = k \):

\[ \tilde{\phi}_k(1, 1, \cdots, 1)(A) \geq \tilde{\phi}_j(a_1, a_2, \cdots, a_j)(A) \]  

(25)

for all \( j \geq 1 \).

It is also of interest to note that when evaluated at \( A = 1 \), the polynomials in (23) or (24) simply give the possible numerators of continued fraction convergents (cf. Hardy and Wright [13]).
4 Bounds for small $n$

In this section, we will obtain the bound in (4) for $4 \leq n \leq 2m + 3$. Recall that

$$m \overset{\text{def}}{=} \left\lfloor \frac{1}{A} \right\rfloor. \tag{26}$$

First, we require the following lemma.

**Lemma 3** Suppose $a_1 + a_2 + \cdots + a_k = n - 2$ with $4 \leq n \leq 2m + 3$ and $a_i \geq 1$ for $i \leq k$, then

$$\phi_k(a_1, a_2, \cdots, a_k) \leq \begin{cases} 
\phi_1(n - 2), & \text{if } k \text{ is odd} \\
\phi_2\left(\frac{n - 2}{2}, \frac{n - 2}{2}\right), & \text{if } k \text{ is even; } n \text{ is even} \\
\phi_2\left(\frac{n - 1}{2}, \frac{n - 3}{2}\right), & \text{if } k \text{ is even; } n \text{ is odd} \\
\phi_1(n - 2), & \text{if } k \text{ is odd} \\
\phi_2\left(\frac{n - 2}{2}, \left\lfloor \frac{n - 3}{2} \right\rfloor\right), & \text{if } k \text{ is even} 
\end{cases}. \tag{27}$$

The following subsidiary lemma is immediate, but will be used frequently below.

**Lemma 4** Let $m = [1/A]$. If $x, y \geq 0$, $n \leq 2m + 3$ and
\[ x + y \leq 2 \left\lfloor \frac{n - 2}{2} \right\rfloor \]

\[ = \begin{cases} 
  n - 2, & \text{if } n \text{ is even} \\
  n - 3, & \text{if } n \text{ is odd}
\end{cases} \tag{28} \]

then

\[ xyA^2 \leq 1. \quad \square \tag{29} \]

**Proof of Lemma 3**

*Case 1: k odd.*

Note that (27) is trivial for \( k = 1 \). Also, employing (23),

\[ \phi_3(a_1, a_2, a_3) = Aa_3\phi_2(a_1, a_2) + \phi_1(a_1) \]

\[ = Aa_3(a_1a_2A^3 + A) + a_1A^2 \]

\[ = (A^2a_3a_1)a_2A^2 + A^2(a_1 + a_3) \]

\[ \leq (a_2 + a_1 + a_3)A^2. \tag{30} \]

The inequality in (30) follows by Lemma 4 with \( S_1 = a_3 \) and \( S_2 = a_1 \).

Now, assume \( K > 3 \) is odd and \( \phi_k(a_1, a_2, \cdots, a_k) \leq (a_1 + a_2 + \cdots + a_k)A^2 \) for \( k = 3, 5, 7, \cdots, K - 2 \). For \( k \) odd and \( k \leq K \), define

\[ J_k = A(a_k + a_{k-2} + \cdots + a_k)\phi_{k-1}(a_1, \cdots, a_{k-1}) + \phi_{k-2}(a_1, \cdots, a_{k-2}). \tag{31} \]

We have
\[ J_k = A(a_K + a_{K-2} + \cdots + a_k)\phi_{K-1}(a_1, \cdots, a_{k-1}) + \phi_{k-2}(a_1, \cdots, a_{k-2}) \]
\[ = A(a_K + a_{K-2} + \cdots + a_k) (Aa_{K-1}\phi_{k-2}(a_1, \cdots, a_{k-2}) + \phi_{k-3}(a_1, \cdots, a_{k-3})) \]
\[ + A\phi_{k-2}\phi_{k-3}(a_1, \cdots, a_{k-3}) + \phi_{k-4}(a_1, \cdots, a_{k-4}) \]
\[ \leq A^2(a_K + a_{K-2} + \cdots + a_k)\phi_{K-1}(a_1 + a_2 + \cdots + a_{k-2}) \]
\[ + A(a_K + a_{K-2} + \cdots + a_k + a_k)\phi_{K-3}(a_1, \cdots, a_{k-3}) \]
\[ + \phi_{k-4}(a_1, \cdots, a_{k-4}) \]
\[ \leq a_{k-1}A^2 + J_{k-2}, \quad (32) \]

where the second to last inequality follows from the induction hypothesis and the final inequality follows by application of Lemma 4 with \( S_1 = a_K + a_{K-2} + \cdots + a_k \) and \( S_2 = a_1 + a_2 + \cdots + a_{k-2} \).

Finally, applying (31), (32) and (23) gives

\[ \phi_K(a_1, \cdots, a_K) = Aa_K\phi_{K-1}(a_1, \cdots, a_{K-1}) + \phi_{K-2}(a_1, \cdots, a_{K-2}) \]
\[ = J_K \]
\[ = \sum_{i=5}^{K} (J_i - J_{i-2}) + J_3 \]
\[ \leq \sum_{i=5}^{K} a_{i-1}A^2 + A(a_K + a_{K-2} + \cdots + a_3)\phi_2(a_1, a_2) + \phi_1(a_1) \]
\[ = A^2(a_{K-1} + a_{K-3} + \cdots + a_4) \]
\[ + A(a_K + a_{K-2} + \cdots + a_3)(a_1a_2A^3 + A) + a_1A^2 \]
\[ = A^2(a_K + a_{K-1} + \cdots + a_4 + a_3 + a_2 + a_1) \]
\[ + A(a_K + a_{K-2} + \cdots + a_3)a_1a_2A^3 \]
\[ \leq (a_K + a_{K-1} + \cdots + a_4 + a_3 + a_2 + a_1)A^2 \]
\[ = \phi_1(n - 2). \quad (33) \]

The first inequality follows from (31) and (32), while the last inequality follows by employing Lemma 4 with \( S_1 = a_K + a_{K-2} + \cdots + a_3 \) and \( S_2 = a_1 \). This proves Lemma 3 for the case that \( k \) is odd.
Case 2: $k$ even.

For $1 \leq j < i \leq K$, $j$ even and $i$ odd, let

$$Q_i = A(a_K + a_{K-2} + \cdots + a_i)\phi_{i-1}(a_1, \cdots, a_{i-1}) + \phi_{i-2}(a_1, \cdots, a_{i-2})$$ \hfill (34)

and

$$L_{i,j} = A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1})\phi_j(a_1, a_2, \cdots, a_j).$$ \hfill (35)

Then,

$$Q_i = A(a_K + a_{K-2} + \cdots + a_i)\phi_{i-1}(a_1, \cdots, a_{i-1}) + \phi_{i-2}(a_1, \cdots, a_{i-2})$$
$$= A(a_K + a_{K-2} + \cdots + a_i) (Aa_{i-1}\phi_{i-2}(a_1, \cdots, a_{i-2}) + \phi_{i-3}(a_1, \cdots, a_{i-3}))$$
$$+ Aa_{i-2}\phi_{i-3}(a_1, \cdots, a_{i-3}) + \phi_{i-4}(a_1, \cdots, a_{i-4})$$
$$= A(a_K + a_{K-2} + \cdots + a_i + a_{i-2})\phi_{i-3}(a_1, \cdots, a_{i-3}) + \phi_{i-4}(a_1, \cdots, a_{i-4})$$
$$+ A^2(a_K + a_{K-2} + \cdots + a_i)a_{i-1}\phi_{i-2}(a_1, \cdots, a_{i-2})$$
$$= Q_{i-2} + L_{i-1,i-2}. \hfill (36)$$

Now, we have

$$L_{i,j} = A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1})\phi_j(a_1, a_2, \cdots, a_j)$$
$$= A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1}) (Aa_j\phi_{j-1}(a_1, \cdots, a_{j-1}) + \phi_{j-2}(a_1, \cdots, a_{j-2}))$$
$$= A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1})\phi_{j-2}(a_1, \cdots, a_{j-2})$$
$$+ A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1})Aa_j\phi_{j-1}(a_1, \cdots, a_{j-1})$$
$$= L_{i,j-2} + A^3a_ia_j(a_K + a_{K-2} + \cdots + a_{i+1})\phi_{j-1}(a_1, \cdots, a_{j-1})$$
$$\leq L_{i,j-2} + A^3a_ia_j(a_K + a_{K-2} + \cdots + a_{i+1})(a_1 + \cdots + a_{j-1})A^2$$
$$\leq L_{i,j-2} + A^3a_ia_j. \hfill (37)$$

The second to last inequality in (37) follows by the result for case 1, while the final
inequality appeals to Lemma 4 with \( S_1 = a_K + a_{K-2} + \cdots + a_{i+1} \) and \( S_2 = a_1 + \cdots + a_{j-1} \) (since \( j < i \)). Hence,

\[
L_{i,i-1} = \sum_{v=4 \atop v \text{ even}}^{i-1} (L_{i,v} - L_{i,v-2}) + L_{i,2} \\
\leq \sum_{v=4 \atop v \text{ even}}^{i-1} A^3a_ia_v + A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1})\phi_2(a_1, a_2) \\
= A^3a_i(a_4 + a_6 + \cdots + a_{i-1}) \\
+ A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1})(a_1a_2A^3 + A) \\
= A^3a_i(a_4 + a_6 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{K-2} + a_K) \\
+ A^2a_i(a_K + a_{K-2} + \cdots + a_{i+1})a_1a_2A^3 \\
\leq A^3a_i(a_4 + a_6 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{K-2} + a_K) + A^3a_ia_2 \\
= A^3a_i(a_2 + a_4 + \cdots + a_{K-2} + a_K). \tag{38}
\]

The last inequality in (38) results from employing Lemma 4 with \( S_1 = a_K + a_{K-2} + \cdots + a_{i+1} \) and \( S_2 = a_1 \).

Now, by employing (36) and (38), we have
\[
\phi_K(a_1, \ldots, a_K) = A a_K \phi_{K-1}(a_1, \ldots, a_{K-1}) + \phi_{K-2}(a_1, \ldots, a_{K-2}) \\
= Q_K \\
= \sum_{i=6}^{K} (Q_i - Q_{i-2}) + Q_4 \\
\leq \sum_{i=6}^{K} (L_{i-1,i-2}) + Q_4 \\
\leq \sum_{i=6}^{K} (A^3 a_{i-1} (a_2 + a_4 + \cdots + a_{K-2} + a_K)) + Q_4 \\
= (a_5 + a_7 + \cdots + a_{K-3} + a_{K-1})(a_2 + a_4 + \cdots + a_{K-2} + a_K)A^3 \\
+ A(a_K + a_{K-2} + \cdots + a_4) \phi_3(a_1, a_2, a_3) + \phi_2(a_1, a_2) \\
= (a_5 + a_7 + \cdots + a_{K-3} + a_{K-1})(a_2 + a_4 + \cdots + a_{K-2} + a_K)A^3 \\
+ A(a_K + a_{K-2} + \cdots + a_4) (A a_3 \phi_2(a_1, a_2) + \phi_1(a_1)) + \phi_2(a_1, a_2). \\
= (a_5 + a_7 + \cdots + a_{K-3} + a_{K-1})(a_2 + a_4 + \cdots + a_{K-2} + a_K)A^3 \\
+ A(a_K + a_{K-2} + \cdots + a_4) \left(A a_3(a_1 a_2 A^3 + A) + a_1 A^2\right) \\
+ a_1 a_2 A^3 + A \\
= (a_5 + a_7 + \cdots + a_{K-3} + a_{K-1})(a_2 + a_4 + \cdots + a_{K-2} + a_K)A^3 \\
+ A^3 a_3 \left(a_1 a_2 A^2(a_K + a_{K-2} + \cdots + a_4) + (a_K + a_{K-2} + \cdots + a_4)\right) \\
+ a_1 A^3 ((a_K + a_{K-2} + \cdots + a_4) + a_2) + A \\
(39)
\]

Finally, applying Lemma 4 to (39) with \(S_1 = a_1\) and \(S_2 = a_K + a_{K-2} + \cdots + a_4\) gives

\[
\phi_K(a_1, \ldots, a_K) \leq (a_5 + a_7 + \cdots + a_{K-3} + a_{K-1})(a_2 + a_4 + \cdots + a_{K-2} + a_K)A^3 \\
+ A^3 a_3 (a_2 + (a_K + a_{K-2} + \cdots + a_4)) \\
+ a_1 A^3 (a_K + a_{K-2} + \cdots + a_4 + a_2) + A \\
= (a_1 + a_3 + a_5 + a_7 + \cdots + a_{K-3} + a_{K-1})(a_2 + a_4 + \cdots + a_{K-2} + a_K)A^3 \\
+ A \\
\leq \left[\frac{n-2}{2}\right] \left[\frac{n-1}{2}\right] A^3 + A \\
= \phi_2 \left(\left[\frac{n-2}{2}\right], \left[\frac{n-1}{2}\right]\right) \\
(40)
\]

This proves Lemma 3. \(\square\)
Proposition 1 (Theorem 1 for small \( n \)). Under the assumptions of Theorem 1, for \( n \leq 3 \),

\[
U_n = \begin{cases} 
A, & \text{if } n = 2 \\
\max(A, A^2), & \text{if } n = 3 \\
\left[\frac{n-2}{2}\right] \left[\frac{n-1}{2}\right] A^3 + A, & \text{if } 4 \leq n \leq 2m + 2 \\
(n - 2)A^2, & \text{if } n = 2m + 3
\end{cases}.
\]

(41)

\[\square\]

**Proof.** The values of \( U_n \) given in (41) for \( n = 2 \) and \( n = 3 \) may be easily verified.

For \( 4 \leq n \leq 2m + 3 \), Lemma 3 implies we need only compare two possible candidates \( \phi \) values for each \( n \). Note that

\[
\phi_2 \left( \left[\frac{n-2}{2}\right], \left[\frac{n-1}{2}\right] \right) - \phi_1 (n-2) = A \left( \left[\frac{n-2}{2}\right] \left[\frac{n-1}{2}\right] A^2 - (n-2)A + 1 \right)
\]

\[
= A \left( \left[\frac{n-2}{2}\right] A - 1 \right) \left( \left[\frac{n-1}{2}\right] A - 1 \right). \tag{42}
\]

Now, for \( n \leq 2m + 2 \),

\[
\left[\frac{n-1}{2}\right] \leq \left[\frac{2m+1}{2}\right]
\]

\[
= \left[\frac{1}{A}\right]
\]

\[
\leq \frac{1}{A}, \tag{43}
\]

and hence the last two factors in (42) are negative, and the product is positive. In
addition, since $1/A - 1 \leq m \leq 1/A$, we have for $n = 2m + 3$ that

$$\left(\left\lfloor \frac{n-2}{2} \right\rfloor A - 1\right) \left(\left\lfloor \frac{n-1}{2} \right\rfloor A - 1\right) = \left(\left\lfloor \frac{2m+1}{2} \right\rfloor A - 1\right) \left(\left\lfloor \frac{2m+2}{2} \right\rfloor A - 1\right) = (mA - 1)((m+1)A - 1) \leq 0. \quad \square$$

(44)

5 Second order recurrence form for $\{U_n\}$

In this section we will establish the bound in (4) for values of $n \geq 2m + 2$, ie. $\{U_i\}_{i \geq 2m+2}$ satisfies a simple second order linear recurrence. First we have the following “bounding” lemma.

**Lemma 5** The sequence $\{U_i\}_{i \geq 2m+2}$ satisfies $U_{2m+2} = m^2 A^3 + A$, $U_{2m+3} = (2m+1)A^2$ and for $n \geq 2m + 4$,

$$U_n \leq AU_{n-1} + U_{n-2}. \quad (45)$$

**Proof.** The values for $U_{2m+2}$ and $U_{2m+3}$ were given in Proposition 1.

First, note that the sequence $\{U_i\}$ must be nondecreasing over values of $i \geq 2$. To see this suppose that (1) is satisfied for $n \leq N$ with $N \geq 3$ and that $U_N = |b_N|$. Now, set $\alpha_{N+1,j} = \alpha_{N,j}$ for $j \leq N - 1$ and $\alpha_{N+1,N} = 0$ then
\[ U_{N+1} \geq |b_{N+1}| \]
\[ = \left| \sum_{k=1}^{N} \alpha_{N+1,k} b_k \right| \]
\[ = \left| \sum_{k=1}^{N-1} \alpha_{N,k} b_k + (0)b_{N} \right| \]
\[ = |b_N| \]
\[ = U_N. \]

Now, suppose that \( \{b_n\} \) is some solution to (1), and \( |b_N| > 0 \) for some \( N \geq 2m + 4 \). By Lemma 1, \( |b_i| \leq |B_i| \) for \( i \geq 1 \). Also, from Lemma 2, \( |B_N| \) can be expressed in the form \( \phi_k(a_1, a_2, \ldots, a_k) \) for some \( k, a_1, a_2, \ldots a_k \geq 1 \) with

\[ \sum_{j=1}^{k} a_j = N - 2. \] (46)

We will consider four cases.

**Case 1:** \( k > 1, a_k > 1 \) and \( a_{k-1} > 1 \). In this case,

\[ \phi_k(a_1, \ldots, a_k) = A\phi_{k-1}(a_1, \ldots, a_{k-2}, (a_k + a_{k-1} - 1)) \]
\[ + \phi_k(a_1, \ldots a_{k-2}, (a_{k-1} - 1), (a_k - 1)) \]
\[ \leq AU_{N-1} + U_{N-2}. \] (47)

Where the inequality follows by induction and (46). To see why the equality in (47) holds, note that for \( k > 3 \)

\[ A\phi_{k-1}(a_1, \ldots, a_{k-2}, (a_k + a_{k-1} - 1)) = A^2(a_k + a_{k-1} - 1)\phi_{k-2}(a_1, \ldots, a_{k-2}) \]
\[ + A\phi_{k-3}(a_1, \ldots, a_{k-3}) \] (48)

and
\[ \phi_k(a_1, \cdots, a_{k-2}, (a_{k-1} - 1), (a_k - 1)) = A(a_k - 1)\phi_{k-1}(a_1, \cdots, a_{k-2}, (a_{k-1} - 1)) \\
+ \phi_{k-2}(a_1, \cdots, a_{k-2}) \\
= A(a_k - 1)(A(a_{k-1} - 1)\phi_{k-2}(a_1, \cdots, a_{k-2}) \\
+ \phi_{k-3}(a_1, \cdots, a_{k-3})) + \phi_{k-2}(a_1, \cdots, a_{k-2}) \\
= (A^2a_k a_{k-1} - A^2a_k - A^2a_{k-1} + A^2) \\
\cdot \phi_{k-2}(a_1, \cdots, a_{k-2}) \\
+ (Aa_k - A)\phi_{k-3}(a_1, \cdots, a_{k-3}) \\
+ \phi_{k-2}(a_1, \cdots, a_{k-2}). \]  

(49)

Combining (48) and (49) and simplifying then gives

\[ A\phi_{k-1}(a_1, \cdots, a_{k-2}, (a_k + a_{k-1} - 1)) + \phi_k(a_1, \cdots, a_{k-2}, (a_{k-1} - 1), (a_k - 1)) \\
= A^2a_k a_{k-1}\phi_{k-2}(a_1, \cdots, a_{k-2}) + Aa_k\phi_{k-3}(a_1, \cdots, a_{k-3}) \\
+ \phi_{k-2}(a_1, \cdots, a_{k-2}) \\
= Aa_k(Aa_{k-1}\phi_{k-2}(a_1, \cdots, a_{k-2}) + \phi_{k-3}(a_1, \cdots, a_{k-3})) \\
+ \phi_{k-2}(a_1, \cdots, a_{k-2}) \\
= Aa_k\phi_{k-1}(a_1, \cdots, a_{k-1}) + \phi_{k-2}(a_1, \cdots, a_{k-2}) \\
= \phi_k(a_1, \cdots, a_k). \]  

(50)

For \( k = 2 \) simply replace \( \phi_{k-2}(a_1, \cdots, a_{k-2}) \) with \( A \) and \( \phi_{k-3}(a_1, \cdots, a_{k-3}) \) with 0 in the argument above. Similarly, for \( k = 3 \) replace \( \phi_{k-3}(a_1, \cdots, a_{k-3}) \) with \( A \).

**Case 2:** \( k > 1, a_k > 1 \) and \( a_{k-1} = 1 \). Here, similarly

\[ \phi_k(a_1, \cdots, a_{k-2}, 1, a_k) = A\phi_{k-1}(a_1, \cdots, a_{k-2}, a_k) \\
+ \phi_{k-2}(a_1, \cdots, a_{k-3}, (a_{k-2} + a_k - 1)) \leq A\varphi_{N-1} + \varphi_{N-2}. \]  

(51)

The arguments to prove (51) are similar to those for (47) and will be omitted.
Case 3: $k > 1$, $a_k = 1$. If $k > 2$, we have directly from (23) that

$$\phi_k(a_1, \cdots, a_{k-1}, 1) = A\phi_{k-1}(a_1, \cdots, a_{k-1}) + \phi_{k-2}(a_1, \ldots, a_{k-2})$$

$$\leq AU_{N-1} + U_{N-a_k-1}.$$  \hfill (52)

Since $\{U_i\}_{i>1}$ is nondecreasing, (52) gives

$$\phi_k(a_1, \cdots, a_{k-1}, 1) \leq AU_{N-1} + U_{N-2}.$$  \hfill (53)

If $k = 2$,

$$\phi_2(a_1, 1) = A\phi_1(a_1) + A$$

$$\leq AU_{N-1} + A,$$  \hfill (54)

and again we have (53) since $U_{N-2} > A$ ($N \geq 4$).

Case 4: $k = 1$. Here, $a_1 = N - 2$. If $N = 2m + 4$, then by Proposition 1,

$$AU_{N-1} + U_{N-2} - \phi_1(a_1) = A(2m + 1)A^2 + m^2 A^3 + A - (2m + 2)A^2$$

$$= A ((m+1)^2 - 2(m+1)A + 1)$$

$$= A ((m+1)A - 1)^2$$

$$\geq 0,$$  \hfill (55)

and hence,

$$\phi_1(a_1) \leq AU_{N-1} + U_{N-2}.$$  \hfill (56)
Meanwhile, if $N \geq 2m + 5$,

\[
\phi_1(a_1) = (N - 2) A^2 \\
= 2A^2 + (N - 4) A^2 \\
= 2A^2 + \phi_1(N - 4) \\
\leq 2A^2 + U_{N-2}.
\]  

(57)

By the definition of $m$, $A \geq 1/(m + 1)$ and hence,

\[
U_{N-1} \geq \phi_1(N - 3) \\
= (N - 3) A^2 \\
\geq (2m + 2) A^2 \\
\geq \frac{2m + 2}{m + 1} A \\
= 2A.
\]  

(58)

From (57) and (58), (55) also holds for $N \geq 2m + 5$, and the proof is complete. □

**Proposition 2** *(Theorem 1 for large $n$). Under the assumptions of Theorem 1*

\[
U_n = \begin{cases} 
  m^2 A^3 + A, & \text{if } n = 2m + 2 \\
  (n - 2) A^2, & \text{if } n = 2m + 3 \\
  AU_{n-1} + U_{n-2}, & \text{if } n \geq 2m + 4
\end{cases}
\]  

(59)

□

In addition to Lemma 5, we will employ the following lemma, which follows directly from (51).
Lemma 6  Let \( \{G_i\} \) be defined by \( G_1 = \phi_2(m, m) \), \( G_2 = \phi_1(2m + 1) \) and \( G_i = \phi_{i-1}(m + 1, 1, 1, \ldots, 1, m + 1) \). Then, \( \{G_i\} \) satisfies \( G_1 = U_{2m+2} \), \( G_2 = U_{2m+3} \) and \( G_i = AG_{i-1} + G_{i-2} \) for \( i \geq 3 \). In addition, each \( G_i \) is of the form \( \phi_{j_i}(a_{i,1}, a_{i,2}, \ldots, a_{i,j_i}) \) with \( \sum_{i=1}^{j_i} a_{i,l} = 2m - 1 + i \), and hence by Lemma 2, \( G_i \leq U_{2m+1+i} \) for \( i \geq 1 \). □

Proof of Proposition 2. By Lemmas 5 and 6,

\[
\begin{align*}
U_{2m+4} &\leq AU_{2m+3} + U_{2m+2} \\
&= AG_2 + G_1 \\
&= G_3 \\
&\leq U_{2m+4},
\end{align*}
\]  

(60)

and hence, \( G_3 = U_{2m+4} \).

Now, suppose \( U_{2m+i+1} = G_i \) for \( i \leq I - 1 \). Then, as in (60),

\[
\begin{align*}
U_{2m+I+1} &\leq AU_{2m+I} + U_{2m+I-1} \\
&= AG_{I-1} + G_{I-2} \\
&= G_I \\
&\leq U_{2m+I+1}.
\end{align*}
\]  

(61)

Hence, \( U_{2m+i+1} = G_i \) for all \( i \geq 1 \). This proves Proposition 2. □

Table 5 displays the dominating polynomials.

27
Table 5
Optimal Polynomials

<table>
<thead>
<tr>
<th>$A$</th>
<th>$[1, \infty)$</th>
<th>$[\frac{1}{2}, 1]$</th>
<th>$[\frac{1}{3}, \frac{1}{2}]$</th>
<th>$[\frac{1}{4}, \frac{1}{3}]$</th>
<th>$[\frac{1}{5}, \frac{1}{4}]$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_2$</td>
<td>$A$</td>
<td>$A$</td>
<td>$A$</td>
<td>$A$</td>
<td>$A$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_3$</td>
<td>$\phi_1(1)$</td>
<td>$A$</td>
<td>$A$</td>
<td>$A$</td>
<td>$A$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_4$</td>
<td>$\phi_2(1, 1)$</td>
<td>$\phi_2(1, 1)$</td>
<td>$\phi_2(1, 1)$</td>
<td>$\phi_2(1, 1)$</td>
<td>$\phi_2(1, 1)$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_5$</td>
<td>$\phi_3(1, 1, 1)$</td>
<td>$\phi_1(3)$</td>
<td>$\phi_2(1, 2)$</td>
<td>$\phi_2(1, 2)$</td>
<td>$\phi_2(1, 2)$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_6$</td>
<td>$\phi_4(1, 1, 1, 1)$</td>
<td>$\phi_2(2, 2)$</td>
<td>$\phi_2(2, 2)$</td>
<td>$\phi_2(2, 2)$</td>
<td>$\phi_2(2, 2)$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_7$</td>
<td>$\phi_5(1, 1, 1, 1)$</td>
<td>$\phi_3(2, 1, 2)$</td>
<td>$\phi_1(5)$</td>
<td>$\phi_2(2, 3)$</td>
<td>$\phi_2(2, 3)$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_8$</td>
<td>$\phi_6(1, 1, 1, 1, 1)$</td>
<td>$\phi_4(2, 1, 1, 2)$</td>
<td>$\phi_2(3, 3)$</td>
<td>$\phi_2(3, 3)$</td>
<td>$\phi_2(3, 3)$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$U_9$</td>
<td>$\phi_7(1, 1, 1, 1, 1, 1)$</td>
<td>$\phi_5(2, 1, 1, 1, 2)$</td>
<td>$\phi_3(3, 1, 3)$</td>
<td>$\phi_1(7)$</td>
<td>$\phi_2(3, 4)$</td>
<td>$\ldots$</td>
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<tr>
<td>$U_{10}$</td>
<td>$\phi_8(1, 1, 1, 1, 1, 1, 1)$</td>
<td>$\phi_6(2, 1, 1, 1, 1, 2)$</td>
<td>$\phi_4(3, 1, 1, 3)$</td>
<td>$\phi_2(4, 4)$</td>
<td>$\phi_2(4, 4)$</td>
<td>$\ldots$</td>
</tr>
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<td>$\vdots$</td>
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<td>$\vdots$</td>
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References


