A Bound for Linear Recurrence Relations with Unbounded Order

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Abstract

This paper considers linear recurrence relations with unbounded order where the coefficients are restricted to intervals of negative real numbers. The optimal inequalities, under the given constraints, exhibit a second order structure.

Key words: Linear Recurrence, Difference Equations, Restricted Coefficients, Optimization, Recurrences with Unbounded Order.


1 Introduction

This paper studies general linear recurrences of the form

\[ b_n = \sum_{k=1}^{n-1} \beta_{n,k} b_k, \] (for \( n \geq 2 \)), (1.1)

where, for some fixed \( A > B \geq 0 \) and \( A \geq 1 \),

\[ \beta_{n,k} \in [-A, -B], \] (1.2)

for \( 1 \leq k \leq n - 1 \) and \( n \geq 2 \). Without loss of generality we will assume that \( b_1 = 1 \).

We are interested, here, in the structure of the bounding sequence \( \{U_n\} \) defined by

\[ U_n = U_n(A, B) \overset{\text{def}}{=} \max \{|b_n| : \{b_i\} \text{ and } \{\beta_{i,j}\} \text{ satisfy (1.1) and (1.2)}\}, \] (1.3)

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for $n \geq 1$.

We will prove the following theorem.

**Theorem 1** With \( \{U_j\} \) defined as in (1.3),

\[
U_n = \begin{cases} 
A, & \text{if } n = 2 \\
\max\{A^2 - B, A - B^2\}, & \text{if } n = 3 \\
A(A^2 - 2B + 1), & \text{if } n = 4 \\
A^4 - 3A^2B + B^2 - B, & \text{if } n = 5 \\
AU_{n-1} + (1 - B)U_{n-2}, & \text{if } n \geq 6
\end{cases}
\]

In [3], an explicit form for \( U_n \) was obtained for the complimentary case of intervals which contain zero, i.e. when (1.2) is replaced by \( \beta_{n,k} \in [-A, B] \) for \( A > B \geq 0 \). The reader is referred to [3] and the references therein for discussion of applications to applicable bounds for reciprocal of power series and inverses of triangular matrices.

We remark that our results and those in [3] leave open the following question regarding subintervals of the negative unit interval.

**Open Question.** What is the value of \( \{U_n\} \) when in place of (1.2), we consider

(1.4) for some fixed \( 1 > C > D > 0 \)

\[
\beta_{n,k} \in [-C, -D],
\]

Linear recurrences as in (1.1) arise in investigation of power series. For related results involving restricted coefficients within that realm see [1], [2], [4], [5], [6], [7] and [8].

## 2 Preliminaries for the proof of Theorem 1

In this section, we will provide a collection of definitions and preliminary results. Suppose \( \{b_i\} \) and \( \{\beta_{i,j}\} \) satisfy (1.1) and (1.2) with \( b_1 = 1 \).

First, for \( n \geq 1 \), let

\[
P_n = \sum_{\substack{1 \leq k \leq n \\ b_k \geq 0}} b_k \quad \text{and} \quad N_n = \sum_{\substack{1 \leq k \leq n \\ b_k < 0}} b_k.
\]
Note that, \( \{P_i\} \) is non-decreasing and \( \{N_i\} \) is non-increasing.

The following elementary bound on \( b_n \) follows from (1.1) and (1.2).

**Lemma 2** For \( n \geq 2 \),

\[
-AP_{n-1} - BN_{n-1} \leq b_n \leq -BP_{n-1} - AN_{n-1} \tag{2.2}
\]

and

\[
-AP_{n-1} - BN_{n-1} < -BP_{n-1} - AN_{n-1}. \tag{2.3}
\]

**PROOF.** By (1.1) and (1.2),

\[
b_n = \sum_{k=1}^{n-1} \beta_{n,k} b_k \geq -A \sum_{1 \leq k \leq n-1, b_k \geq 0} b_k - B \sum_{1 \leq k \leq n-1, b_k < 0} b_k = -AP_{n-1} - BN_{n-1}.
\]

The upper bound in (2.2) follows by a similar argument and (2.3) follows from

\[
-BP_{n-1} - AN_{n-1} - (-AP_{n-1} - BN_{n-1}) = (A - B)(P_{n-1} - N_{n-1}) > 0.
\]

**Theorem 3** We have \( U_2 = A \) and \( U_3 = \max\{A^2 - B, A - B^2\} \).

**PROOF.** By (1.1), we have \( b_1 = 1 \) and

\[
b_2 = \beta_{2,1} b_1 = \beta_{2,1} \in [-A, -B].
\]

Thus, we have

\[
U_2 = A
\]

and

\[
\begin{cases}
P_2 = b_1 = 1 \\
N_2 = b_2 \in [-A, -B]
\end{cases}
\]

(2.4)

By Lemma 2, \( -AP_2 - BN_2 \leq b_3 \leq -BP_2 - AN_2 \) and

\[
-AP_2 - BN_2 \geq -A(1) - B(-B) = B^2 - A \\
-BP_2 - AN_2 \leq -B(1) - A(-A) = A^2 - B > 0.
\]
Hence $B^2 - A \leq b_3 \leq A^2 - B$.

If $B^2 - A \geq 0$, i.e. $A - B^2 \leq 0$, then,

$$U_3 = A^2 - B = \max\{A^2 - B, A - B^2\}.$$ 

If $B^2 - A < 0$, then

$$U_3 = \max\{|A^2 - B|, |B^2 - A|\} = \max\{A^2 - B, A - B^2\}.$$ 

**Note 1** The value of $A - B^2$ is greater than that of $A - B^2$ only in the small region

$$\{(A, B) : 0 \leq B \leq 1 \text{ and } 1 \leq A \leq \frac{1}{2}(1 + \sqrt{1 + 4B - 4B^2})\}. \quad (2.5)$$

This region is shown in Figure 1.

![Fig. 1. The region in which $A - B^2$ is greater than $A^2 - B$](image)

**Lemma 4** If $B \leq 1$, $n \geq 2$ and

$$-AP_{n-1} - BN_{n-1} \leq 0 \leq -BP_{n-1} - AN_{n-1}, \quad (2.6)$$

then

$$-AP_n - BN_n \leq 0 \leq -BP_n - AN_n. \quad (2.7)$$

**PROOF.** The result is trivial for $B = 0$.

Suppose $b_n \geq 0$. Then $N_n = N_{n-1}$ and by Lemma 2, we have $0 \leq b_n \leq -BP_{n-1} - AN_{n-1}$ and

$$P_{n-1} \leq P_n \leq P_{n-1} - BP_{n-1} - AN_{n-1} = (1 - B)P_{n-1} - AN_{n-1}. $$


Hence,

\[ AP_n + Bn \geq AP_{n-1} + Bn_{n-1} \geq 0 \]

and, since \(0 \leq B \leq 1\),

\[
BP_n + AN_n \leq B[(1 - B)P_{n-1} - AN_{n-1}] + AN_{n-1}
\]
\[
= (1 - B)(BP_{n-1} + AN_{n-1})
\]
\[
\leq 0.
\]

The case for \(b_n < 0\) follows by a similar argument.

**Corollary 5** If \(B \leq 1\), then

\[-AP_n - BN_n \leq 0 \leq -BP_n - AN_n \text { for } n \geq 2. \tag{2.8} \]

**Proof.** First by (1.1) and (1.2), \(b_1 = 1\) and \(b_2 = \beta_{2,1} \in [-A, -B]\). Thus, we have \(P_2 = b_1 = 1\) and \(N_2 = b_2 \in [-A, -B]\), and hence,

\[-AP_2 - BN_2 \leq A(B - 1) \leq 0 \leq B(A - 1) \leq -BP_2 - AN_2. \]

Equation (2.8) follows by induction via Lemma 4.

**Definition 6** Let \(\{\tilde{b}_i\}_{i=1}^n\) be an instance of \(\{b_i\}_{i=1}^n\) for which \(|b_n|\) attains the bound \(U_n\). We denote the \(\{\beta_{i,j}\}, \{P_i\}\) and \(\{N_i\}\) corresponding to this instance by \(\{\tilde{\beta}_{i,j}\}, \{\tilde{P}_i\}\) and \(\{\tilde{N}_i\}\), respectively.

**Note 1** The sequence \(\{\tilde{b}_i\}_{i=1}^n\) is defined such that \(|\tilde{b}_n| = U_n\), it is not required that \(|\tilde{b}_i| = U_i\) for \(1 \leq i \leq n - 1\).

**Note 2** Whenever \(\tilde{b}_i\) is used in this paper, we always assume that it is coming from \(\{\tilde{b}_i\}_{i=1}^n\) with \(|\tilde{b}_n| = U_n\). The value of \(n\) should be clear from the context.

**Lemma 7** For \(n \geq 4\), \(\tilde{b}_n\) and \(\tilde{b}_{n-1}\) must have alternating signs and, specifically, we either have

\[
\tilde{b}_n = A(B - 1)\tilde{P}_{n-2} + (A^2 - B)\tilde{N}_{n-2} < 0 \quad \text{and} \quad \tilde{b}_{n-1} = -B\tilde{P}_{n-2} - A\tilde{N}_{n-2} \geq 0 \tag{2.9}
\]

or

\[
\tilde{b}_n = (A^2 - B)\tilde{P}_{n-2} + A(B - 1)\tilde{N}_{n-2} > 0 \quad \text{and} \quad \tilde{b}_{n-1} = -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} < 0. \tag{2.10}
\]

**Proof.** From (2.2),

\[-A\tilde{P}_{n-2} - B\tilde{N}_{n-2} \leq \tilde{b}_{n-1} \leq -B\tilde{P}_{n-2} - A\tilde{N}_{n-2}.\]
Depending on the signs of \(-A\hat{P}_{n-2} - B\hat{N}_{n-2}\) and \(-B\hat{P}_{n-2} - A\hat{N}_{n-2}\), there are three cases to consider.

**Case 1** \((-A\hat{P}_{n-2} - B\hat{N}_{n-2} \leq 0\) and \(-B\hat{P}_{n-2} - A\hat{N}_{n-2} \geq 0\).

By (1.1),

\[
\tilde{b}_n = \sum_{k=1}^{n-2} \tilde{\beta}_{n,k} \tilde{b}_k + \tilde{\beta}_{n,n-1} \tilde{b}_{n-1} \\
\geq -A \sum_{1 \leq k \leq n-2} \tilde{b}_k - B \sum_{1 \leq k \leq n-2} \tilde{b}_k + \tilde{\beta}_{n,n-1} \tilde{b}_{n-1} \\
\geq (-A\hat{P}_{n-2} - B\hat{N}_{n-2}) - A(-B\hat{P}_{n-2} - A\hat{N}_{n-2}) \\
= A(B - 1)\hat{P}_{n-2} + (A^2 - B)\hat{N}_{n-2}.
\]

Equality holds for (2.11) when \(\tilde{b}_{n-1} = -B\hat{P}_{n-2} - A\hat{N}_{n-2} \geq 0\) and \(\tilde{\beta}_{n,n-1} = -A\).

Similarly,

\[
\tilde{b}_n = \sum_{k=1}^{n-2} \tilde{\beta}_{n,k} \tilde{b}_k + \tilde{\beta}_{n,n-1} \tilde{b}_{n-1} \\
\leq -B \sum_{1 \leq k \leq n-2, \tilde{b}_k \geq 0} \tilde{b}_k - A \sum_{1 \leq k \leq n-2, \tilde{b}_k < 0} \tilde{b}_k + \tilde{\beta}_{n,n-1} \tilde{b}_{n-1} \\
\leq (-B\hat{P}_{n-2} - A\hat{N}_{n-2}) - A(-B\hat{P}_{n-2} - B\hat{N}_{n-2}) \\
= (A^2 - B)\hat{P}_{n-2} + A(B - 1)\hat{N}_{n-2}.
\]

Equality holds for (2.12) when \(\tilde{b}_{n-1} = -B\hat{P}_{n-2} - B\hat{N}_{n-2} < 0\) and \(\tilde{\beta}_{n,n-1} = -A\).

Note, by (2.3), the RHS in (2.11) is negative and the RHS in (2.12) is positive, and hence we either have (2.9) or (2.10).

**Case 2** \((-A\hat{P}_{n-2} - B\hat{N}_{n-2} \geq 0\) and \(-B\hat{P}_{n-2} - A\hat{N}_{n-2} > 0\).

By a similar argument to that in Case 1, we either have

\[
\tilde{b}_{n-1} = -B\hat{P}_{n-2} - A\hat{N}_{n-2} > 0 \\
\tilde{b}_n = A(B - 1)\hat{P}_{n-2} + (A^2 - B)\hat{N}_{n-2} \\
= -A\hat{P}_{n-2} - B\hat{N}_{n-2} - A(-B\hat{P}_{n-2} - A\hat{N}_{n-2}) < 0
\]

or
\[ \tilde{b}_{n-1} = -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} > 0 \quad (2.15) \]
\[ \tilde{b}_n = B(A - 1)\tilde{P}_{n-2} + (B^2 - A)\tilde{N}_{n-2} > 0 \quad (2.16) \]

Note that the RHS in (2.14) is more negative than that of (2.16) and thus has a larger absolute value if the latter is non-positive.

Suppose the RHS of (2.16) is positive. We have
\[ \tilde{N}_{n-2} < -\frac{A}{B}\tilde{P}_{n-2} \]
and by Corollary 5, \( B > 1 \).

Hence,
\[ |A(B - 1)\tilde{P}_{n-2} + (A^2 - B)\tilde{N}_{n-2}| - |B(A - 1)\tilde{P}_{n-2} + (B^2 - A)\tilde{N}_{n-2}| \]
\[ = -A(B - 1)\tilde{P}_{n-2} - (A^2 - B)\tilde{N}_{n-2} - B(A - 1)\tilde{P}_{n-2} - (B^2 - A)\tilde{N}_{n-2} \]
\[ = -[A(B - 1) + B(A - 1)]\tilde{P}_{n-2} - [A(A - 1) + B(B - 1)]\tilde{N}_{n-2} \]
\[ \geq -[A(B - 1) + B(A - 1)]\tilde{P}_{n-2} - [A(A - 1) + B(B - 1)] \left[ -\frac{A}{B}\tilde{P}_{n-2} \right] \]
\[ = \frac{(A^2 - B^2)(A - 1)}{B} \tilde{P}_{n-2} \geq 0, \]
and thus (2.15) and (2.16) do not hold.

**Case 3** \((-A\tilde{P}_{n-2} - B\tilde{N}_{n-2} < 0 \text{ and } -B\tilde{P}_{n-2} - A\tilde{N}_{n-2} < 0\)).

By a similar argument to that in Case 2, we have,
\[ \tilde{b}_{n-1} = -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} < 0 \quad (2.17) \]
\[ \tilde{b}_n = A(B - 1)\tilde{N}_{n-2} + (A^2 - B)\tilde{P}_{n-2} > 0 \quad (2.18) \]

**Definition 8** Let \( \{\hat{b}_n\} \) be a special case of \( \{b_n\} \) obtained by setting
\[ \hat{\beta}_{i,j} = \hat{\beta}_{i,j} = \begin{cases} -B, & \text{if } i \equiv j \mod (2) \\ -A, & \text{if } i \not\equiv j \mod (2) \end{cases} \quad (2.19) \]

**Lemma 9** We have
\[ \hat{b}_1 = 1, \quad (2.20) \]
\[ \hat{b}_2 = -A, \quad (2.21) \]
\[ \hat{b}_3 = A^2 - B, \quad (2.22) \]
\[ \hat{b}_4 = -A(A^2 - 2B + 1), \quad \text{and} \quad (2.23) \]
\[ \hat{b}_5 = A^4 - 3A^2B + B^2 - B. \quad (2.24) \]
PROOF. By direct computation.

Note 1 Note that $U_1 = \hat{b}_1$ and $U_2 = -\hat{b}_2$. If $(A, B)$ is not in (2.5), we also have $U_3 = \hat{b}_3$.

Lemma 10 For $n \geq 4$,

$$\hat{b}_n = -A\hat{b}_{n-1} + (1 - B)\hat{b}_{n-2}. \quad (2.25)$$

PROOF. By Definition 8 and (1.1),

$$\hat{b}_n = \sum_{i=1}^{n-3} \hat{\beta}_{n,i} \hat{b}_i + \hat{\beta}_{n,n-2} \hat{b}_{n-2} + \hat{\beta}_{n,n-1} \hat{b}_{n-1}$$

$$= \sum_{i=1}^{n-3} \hat{\beta}_{n-2,i} \hat{b}_i + (-B)\hat{b}_{n-2} + (A)\hat{b}_{n-1}$$

$$= \hat{b}_{n-2} + (-B)\hat{b}_{n-2} + (-A)\hat{b}_{n-1}$$

$$= (1 - B)\hat{b}_{n-2} - A\hat{b}_{n-1}.$$ 

Lemma 11 $\{\hat{b}_n\}$ is alternating.

PROOF. From Lemma 9, it is not hard to see that the first three terms of $\{\hat{b}_n\}$ are alternating and for $n=3$, we have

$$\hat{b}_3 = \begin{cases} -BP_{n-1} - AN_{n-1} > 0, & \text{if } n \text{ is odd} \\ -AP_{n-1} - BN_{n-1} < 0, & \text{if } n \text{ is even} \end{cases}. \quad (2.26)$$

Assume (2.26) is true for $3 \leq n \leq k - 1$ and $\{\hat{b}_n\}_{n=1}^{k-1}$ is alternating. Without loss of generality, suppose $k - 1$ is odd. Then $k - 1$ is even, and thus we have

$$\hat{b}_{k-1} = -AP_{k-2} - BN_{k-2} < 0.$$ 

Definition 8 and (1.1) give
\[ \hat{b}_k = \sum_{1 \leq i \leq k-1} \hat{\beta}_{k,i} \hat{b}_i + \sum_{1 \leq i \leq k-1} \hat{\beta}_{k,i} \hat{b}_i \]
\[ = -B \sum_{1 \leq i \leq k-1} \hat{b}_i - A \sum_{1 \leq i \leq k-1} \hat{b}_i \]
\[ = -BP_{k-1} - AN_{k-1} \]
\[ = -BP_{k-2} - A(N_{k-2} + \hat{b}_{k-1}) \]
\[ = -BP_{k-2} - A[(N_{k-2} - AP_{k-2} - BN_{k-2}] \]
\[ = (-BP_{k-2} - AN_{k-2}) - A(-AP_{k-2} - BN_{k-2}), \]

which, by the induction assumption and (2.3), is positive.

The case when \( k \) is even is similar, and the proof is finished by mathematical induction.

3 When \( B \leq 1 \) and \( n \geq 4 \)

In this section, we consider the case where \( B \leq 1 \) and \( n \geq 4 \).

**Lemma 12** When \( B \leq 1 \) and \( n \geq 4 \), \( \tilde{b}_n, \tilde{b}_{n-1} \) and \( \tilde{b}_{n-2} \) must have alternating signs, specifically we have one of the following two possibilities.

1. \((\text{sgn}(\tilde{b}_{n-2}, \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n)) = (+, -, +) \) and

\[ \tilde{b}_{n-2} = -B\tilde{P}_{n-3} - A\tilde{N}_{n-3} \geq 0, \quad (3.1) \]
\[ \tilde{b}_{n-1} = A(B - 1)\tilde{P}_{n-3} + (A^2 - B)\tilde{N}_{n-3} \]
\[ = -A\tilde{P}_{n-3} - B\tilde{N}_{n-3} - A(-B\tilde{P}_{n-3} - A\tilde{N}_{n-3}) < 0, \quad (3.2) \]

and

\[ \tilde{b}_n = (A^2 - B)(1 - B)\tilde{P}_{n-3} - A(A^2 - 2B + 1)\tilde{N}_{n-3} \]
\[ = -A\tilde{b}_{n-1} + (1 - B)\tilde{b}_{n-2} > 0; \quad (3.4) \]

2. \((\text{sgn}(\tilde{b}_{n-2}, \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n)) = (-, +, -) \) and

\[ \tilde{b}_{n-2} = -A\tilde{P}_{n-3} - B\tilde{N}_{n-3} < 0, \quad (3.5) \]
\[ \tilde{b}_{n-1} = (A^2 - B)\tilde{P}_{n-3} + A(B - 1)\tilde{N}_{n-3} \]
\[ = -B\tilde{P}_{n-3} - A\tilde{N}_{n-3} - A(-A\tilde{P}_{n-3} - B\tilde{N}_{n-3}) > 0, \quad (3.6) \]

and

\[ \tilde{b}_n = -A(A^2 - 2B + 1)\tilde{P}_{n-3} + (A^2 - B)(1 - B)\tilde{N}_{n-3} \]
\[ = -A\tilde{b}_{n-1} + (1 - B)\tilde{b}_{n-2} < 0. \quad (3.8) \]
**Proof.** From Lemma 7, we know that $\tilde{b}_n$ and $\tilde{b}_{n-1}$ must have alternating signs.

If $(\text{sgn}(\tilde{b}_{n-2}), \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n)) = (+, -, +)$, by (2.10),

$$|\tilde{b}_n| = (A^2 - B)\tilde{P}_{n-2} + A(B - 1)\tilde{N}_{n-2}$$
$$= (A^2 - B)(\tilde{P}_{n-3} + \tilde{b}_{n-2}) + A(B - 1)\tilde{N}_{n-3}. \quad (3.9)$$

If $(\text{sgn}(\tilde{b}_{n-2}), \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n)) = (+, +, -)$, by (2.9),

$$|\tilde{b}_n| = -[A(B - 1)\tilde{P}_{n-2} + (A^2 - B)\tilde{N}_{n-2}]$$
$$= -[A(B - 1)(\tilde{P}_{n-3} + \tilde{b}_{n-2}) + (A^2 - B)\tilde{N}_{n-3}]. \quad (3.10)$$

In both cases, a larger value for $\tilde{b}_{n-2}$ gives a larger value for $\tilde{b}_n$. Thus from (2.2), $\tilde{b}_{n-2} = -B\tilde{P}_{n-3} - A\tilde{N}_{n-3}$. Substituting this into (3.9) and (3.10), and taking the difference gives

$$(3.9) - (3.10) = [(A - 1)(A + B)][\tilde{P}_{n-3} + (-B\tilde{P}_{n-3} - A\tilde{N}_{n-3}) + \tilde{N}_{n-3}]$$
$$= [(A - 1)(A + B)][(1 - B)\tilde{P}_{n-3} + (1 - A)\tilde{N}_{n-3}]$$
$$\geq 0,$$

$(\text{sgn}(\tilde{b}_{n-2}), \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n))$ cannot be $(+, +, -)$,

and if $(\text{sgn}(\tilde{b}_{n-2}), \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n)) = (+, -, +)$,

then $\tilde{b}_{n-2}, \tilde{b}_{n-1}$ and $\tilde{b}_n$ are as in (3.1), (3.2) and (3.3), respectively.

A similar proof shows that $(\text{sgn}(\tilde{b}_{n-2}), \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n))$ cannot be $(-, -, +)$,

and if $(\text{sgn}(\tilde{b}_{n-2}), \text{sgn}(\tilde{b}_{n-1}), \text{sgn}(\tilde{b}_n)) = (-, +, -)$

then $\tilde{b}_{n-2}, \tilde{b}_{n-1}$ and $\tilde{b}_n$ are as in (3.5), (3.6) and (3.7), respectively.

Theorem 13 When $B \leq 1$,

$$U_4 = A(A^2 - 2B + 1). \quad (3.11)$$

**Proof.** By definition, $\tilde{b}_2 \leq 0$. Thus by Lemma 12, $(\text{sgn}(\tilde{b}_2), \text{sgn}(\tilde{b}_3), \text{sgn}(\tilde{b}_4))$ must be $(-, +, -)$. By (3.7), we have

$$U_4 = -[-A(A^2 - 2B + 1)\tilde{P}_1 + (A^2 - B)(1 - B)\tilde{N}_1]$$
$$= -[-A(A^2 - 2B + 1)(1) + (A^2 - B)(1 - B)(0)]$$
$$= A(A^2 - 2B + 1).$$

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Theorem 14  When $B \leq 1$, $U_n = |\tilde{b}_n|$, for $n \geq 4$.

**PROOF.** From Lemma 9 and Theorem 13, $U_4 = |\tilde{b}_4|$.

Suppose $U_i = |\tilde{b}_i|$ for $4 \leq i \leq n - 1$.

From Lemma 12, and more specifically (3.4) and (3.8), we know that

$$\tilde{b}_n = -A\tilde{b}_{n-1} + (1 - \tilde{B})\tilde{b}_{n-2},$$

(3.12)

where $\tilde{b}_{n-2}$ is of the same sign as and $\tilde{b}_{n-1}$ is of opposite sign to $\tilde{b}_n$. Hence we have

$$U_n = |\tilde{b}_n| = A|\tilde{b}_{n-1}| + (1 - B)|\tilde{b}_{n-2}| \leq AU_{n-1} + (1 - B)U_{n-2}. \quad (3.13)$$

Now from (3.13), and Lemmas 10 and 11, we have

$$U_n \geq |\tilde{b}_n| = A|\tilde{b}_{n-1}| + (1 - B)|\tilde{b}_{n-2}| = AU_{n-1} + (1 - B)U_{n-2} \geq U_n.$$  

Thus, we have $U_n = |\tilde{b}_n|$. The proof is completed by induction.

This concludes the proof of Theorem 1 when $B \leq 1$ and $n \geq 4$. We now turn to the proof of Theorem 1 when $B > 1$ and $n \geq 4$.

4 When $B > 1$ and $n \geq 4$

In this section, we consider the case when $B > 1$ and $n \geq 4$.

For convenience, when $B > 1$, we modify the original problem to allow $b_1$ to be 1 or $-1$. In switching $b_1$ from 1 to $-1$, while keeping the values of all $\beta_{i,j}$ unchanged, only the signs of every term in $\{b_n\}$ change but the original moduli are preserved. Hence this modified problem yields the same $U_n$ as the original problem and in the modified problem $U_n = \max\{b_n : b_n \geq 0\}$.

Lemma 15 If $B > 1$ and $\tilde{b}_n \geq 0$, where $n \geq 4$, we have

$$\tilde{b}_n = \begin{cases} (1 - B)\hat{b}_m \tilde{P}_{n-m} + \hat{b}_{m+1} \tilde{N}_{n-m}, & \text{for } 0 < m < n \text{ and } m \text{ is odd} \\ \hat{b}_{m+1} \tilde{P}_{n-m} + (1 - B)\hat{b}_m \tilde{N}_{n-m}, & \text{for } 0 < m < n \text{ and } m \text{ is even} \end{cases}, \quad (4.1)$$
**PROOF.** From Lemma 7, we know (4.1) is true for $m=1$. Suppose (4.1) is true for $1 \leq m \leq k - 1$.

If $k$ is odd, then from the induction assumption, we have

$$\tilde{b}_n = \hat{b}_k \tilde{P}_{n-k+1} + (1-B)\hat{b}_{k-1}\tilde{N}_{n-k+1}.$$  

Note also that

$$\begin{align*}
\tilde{P}_{n-k+1} &= \tilde{P}_{n-k} + \tilde{b}_{n-k+1} & \text{and} \quad \tilde{N}_{n-k+1} = \tilde{N}_{n-k}, & \text{if } b_{n-k+1} \geq 0 \\
\tilde{P}_{n-k+1} &= \tilde{P}_{n-k} & \text{and} \quad \tilde{N}_{n-k+1} = \tilde{N}_{n-k} + \tilde{b}_{n-k+1}, & \text{if } b_{n-k+1} < 0.
\end{align*}$$

From Lemma 11 and the condition $B > 1$, we know that $\hat{b}_k > 0$ and $(1-B)\hat{b}_{k-1} > 0$. So a more positive $\tilde{P}_{n-k+1}$ would give a larger $|\tilde{b}_n|$ than a more negative $\tilde{N}_{n-k+1}$. Thus, we have $b_{n-k+1} = -BP_{n-k} - AN_{n-k}$ and so we have

$$\begin{align*}
\tilde{b}_n &= \hat{b}_k (\tilde{P}_{n-k} + \tilde{b}_{n-k+1}) + (1-B)\hat{b}_{k-1}\tilde{N}_{n-k} \\
&= \hat{b}_k (\tilde{P}_{n-k} - BP_{n-k} - AN_{n-k}) + (1-B)\hat{b}_{k-1}\tilde{N}_{n-k} \\
&= (1-B)\hat{b}_k \tilde{P}_{n-k} + [-A\hat{b}_k + (1-B)\hat{b}_{k-1}]\tilde{N}_{n-k} \\
&= (1-B)\hat{b}_k \tilde{P}_{n-k} + \hat{b}_{k+1}\tilde{N}_{n-k}.
\end{align*}$$

The argument when $k$ is even is very similar and the proof is completed by mathematical induction.

**Theorem 16** For $n \geq 4$, if $B > 1$, $U_n = |\tilde{b}_n|$.

**PROOF.** In the modified problem, we only need to consider the case when $\tilde{b}_n \geq 0$. From (4.1), we have

$$\tilde{b}_n = \begin{cases} 
\hat{b}_n P_1 + (1-B)\hat{b}_{n-1} N_1, & \text{if } n \text{ is odd} \\
(1-B)\hat{b}_{n-1} P_1 + \hat{b}_n N_1, & \text{if } n \text{ is even} 
\end{cases} \quad (4.2)$$

Note that $(P_1, N_1)$ can only be $(1,0)$ or $(0,-1)$.

If $n$ is odd, then $\tilde{b}_n > 0$ and $(1-B)\tilde{b}_{n-1} > 0$, so we would choose $(P_1, N_1)$ to be $(1,0)$, and hence $\tilde{b}_n = \hat{b}_n$.

If $n$ is even, then $\tilde{b}_n < 0$ and $(1-B)\tilde{b}_{n-1} < 0$, so we would choose $(P_1, N_1)$ to be $(0,-1)$, and hence $\tilde{b}_n = \hat{b}_n$.

So combining both cases, we have $|U_n| = |\tilde{b}_n|$.
5 Conclusion

Theorem 1 now follows from Theorems 3, 13, 14 and 15 and Lemmas 9 and 10.

We restate that our results leave open the case of subintervals of the negative unit interval. The results herein and in [3] give that the sequence in (1.3) eventually satisfies a second order recurrence for all other intervals. We conjecture that the same holds for the remaining cases. But it appears that novel techniques will be needed.

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