Bounds for Fourth-Order $[0, 1]$ Difference Equations

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Abstract

This note examines bounds for fourth order linear difference equations with coefficients restricted to the unit interval. It is shown that all solutions are of order strictly less than $(3/2)^n$. The bound is shown to be nearly best possible. Applications to zero-one banded matrices are also discussed.

Key words: Explicit bounds, Fourth-order difference equations, Restricted coefficients, Growth rates, Nonconstant coefficients.


1 Introduction

This paper studies bounds for fourth-order linear difference equations with coefficients restricted to the interval $[0, 1]$. In particular, suppose that for $n \geq 1$, we have

$$b_n + \alpha_{n,n-1}b_{n-1} + \alpha_{n,n-2}b_{n-2} + \alpha_{n,n-3}b_{n-3} + \alpha_{n,n-4}b_{n-4} = 0$$

(1)

where $\{\alpha_{i,j}\}$ satisfies

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\[ \alpha_{i,j} \in [0, 1], \quad (2) \]

for \( i \geq 1 \) and \(-3 \leq j \leq i - 1 \), and \( b_{-3}, b_{-2}, b_{-1} \) and \( b_0 \) are some initial starting values.

Behavior of solutions to higher order linear equations as in (1) have been studied in the past (c.f. [6], [7], [14], [18], [19], [20], [21] and [26]).

Linear recurrences with varying or random coefficients have been studied by many previous authors. A partial survey of such literature contains Viswanath [22] and [23], Viswanath and Trefethen [24], Embree and Trefethen [8], Wright and Trefethen [25], Mallik [12], Popenda [17], Kittapa [11], and Odlyzko [15].

While of interest from a theoretical standpoint, bounds for recurrences such as those in (1) can be useful in a range of applications through connections to triangular matrix equations. In Section 4, below, we discuss an application to (0, 1) banded matrices.

Bounds for second order recurrences with restricted coefficients have been studied in [5].

We are interested, here in bounds on the rate of increase of \( \{ |b_i| \} \), for any solution \( \{ b_i \} \) of (1), subject to the constraint in (2). In particular we will consider \( \{ U_i \} \) defined by

\[
U_n \overset{\text{def}}{=} \max \{|b_n| : \{ b_i \}, \{ \alpha_{i,j} \} \text{ satisfies (1), (2), and } (b_{-3}, b_{-2}, b_{-1}, b_0) = (0, 0, 0, 1) \},
\]

for \( n \geq 1 \), and will prove the following.

**Theorem 1** We have

\[
U_n < (3/2)^n,
\]

for all \( n \).

The inequality in (4) then leads to the following general theorem.

**Theorem 2** Suppose that \( \{ b_i \} \) and \( \{ \alpha_{i,j} \} \) satisfy (1) and (2). Then,

\[
|b_n| < (|b_{-3}| + |b_{-2}| + |b_{-1}| + |b_0|)(3/2)^n.
\]

for all \( n \geq 0 \).
In Section 4, below, we will construct a sequence for which the growth rate is \( \rho_0 = 1.496 \) and hence the rate in (1) is nearly optimal. Computations suggest that \( \rho_0 \) is actually the best possible rate.

The remainder of the paper proceeds as follows. Section 2 contains some preliminary results and notation. In Section 3, we prove Theorems 1 and 2, while Section 4 includes discussion of optimality as well as applications to bounds for inverses of banded matrices.

2 Preliminary results and notation

Suppose that \((P, N)\) is a partition of the set \( \mathbb{N} = \{0, 1, 2, 3, \cdots \} \). Define \( \{B_i\}_{i \geq 0} \), recursively in \( n \), from \( P \) (and \( N \)) via

\[
B_0 = \begin{cases} 
-1 & n \in P \\
0 & n \in N 
\end{cases}
\]

for \( n \geq 1 \).

Simple induction with (6) will show that \( B_n < 0 \) if and only if \( n \in N \).

Let \( I_n = \{0, 1, 2, \cdots, n\}, P_n = P \cap I_n \) and \( N_n = N \cap I_n \). We will denote the partition \((P_n, N_n)\) of \( I_n \) by \((P, N)\). Since \( B_n \) is a function of \( P \), we will sometimes denote \( B_n \) by \( B_n(P) \).

Now, Suppose that \( \{b_i\} \) and \( \{\alpha_{i,j}\} \) satisfy (1) and (2) with \((b_{-3}, b_{-2}, b_{-1}, b_0) \in \{(0, 0, 0, 1), (0, 0, 0, -1)\} \).

The following lemma reduces the problem of bounding \( |b_n| \) to a comparison of the individual values of \( |B_n| \) for the at most \( 2^{n+1} \) possible partitions into two sets of \( I_n \).

**Lemma 1** If \( B_0 = b_0 \), \( P = \{i \geq 0 : b_i \geq 0\} \) and \( N = \{i \geq 0 : b_i < 0\} \), then

\[
|b_i| \leq |B_i(P)|,
\]

for all \( i \geq 0 \).
For a fixed $I$ through $\pi_\alpha$.

Proof. First, note that under the inherent assumptions, $|b_0| = 1 = |B_0|$ and $|b_1| = -\alpha_{1,0} \leq 1 = |B_1|$. We shall prove the lemma by induction. Suppose that $N > 0$ and that (7) is satisfied for all $i \leq N - 1$. Now, assume that $N \in \mathcal{P}$. Then,

$$b_N = \sum_{1 \leq i \leq 4} \alpha_{N,N-i} b_{N-i} \leq - \sum_{1 \leq i \leq 4, N-i \in \mathcal{N}} b_{N-i} = \sum_{1 \leq i \leq 4, N-i \in \mathcal{N}} |b_{N-i}|$$

$$\leq \sum_{1 \leq i \leq 4, N-i \in \mathcal{N}} |B_{N-i}| = - \sum_{1 \leq i \leq 4, N-i \in \mathcal{N}} B_{N-i} = B_N,$$

(8)

where the first inequality follows from (2) and the second from an application of the induction hypothesis.

An analogous argument works when $n \in \mathcal{N}$. □

It will also be useful to have the following symmetry result.

**Lemma 2** For a fixed $n > 1$, define $\mathcal{N}^*$ and $\mathcal{P}^*$ via $i \in \mathcal{N}^*$ ($i \in \mathcal{P}^*$) if and only if $n - i \in \mathcal{N}$ ($n - i \in \mathcal{P}$) for $0 \leq i \leq n$ then $|B_n(\mathcal{P}^*)| = |B_n(\mathcal{P})|$.

Proof. This follows from the combinatorial fact (see Lemma 2 in [2], with $k = 4$) that $|B_n(\mathcal{P})|$ is equal to the number of “paths” $\pi = (p_1, p_2 \cdots, p_t)$ from 1 to $n$ (i.e. $p_1 = 1$ and $p_t = n$) in $\mathcal{I}_n$ such that $0 < p_{i+1} - p_i \leq 4$, and either $i \in \mathcal{N}$ and $i + 1 \in \mathcal{P}$ or $i \in \mathcal{P}$ and $i + 1 \in \mathcal{N}$, for all $1 \leq i \leq t - 1$. □

We will say that $(\mathcal{P}, \mathcal{N})_n$ has a positive (negative) semicycle of length $r \geq 1$ (an $r$-cycle) in $\mathcal{I}_n$ beginning at $j \geq 1$, if $i \in \mathcal{P}$ ($i \in \mathcal{N}$) for $j \leq i \leq j + r - 1 \leq n$ and either $j - 1 \in \mathcal{N}$ ($j - 1 \in \mathcal{P}$) or $j = 0$ and either $j + r \in \mathcal{N}$ ($j + r \in \mathcal{P}$) or $j + r - 1 = n$. If there exists a $j$ such that $(\mathcal{P}, \mathcal{N})_n$ has a semicycle of length $r$ (in $\mathcal{I}_n$) beginning at $j$ we will say that $(\mathcal{P}, \mathcal{N})_n$ contains a semicycle of length $r$.

We have the following lemma.

**Lemma 3** If $(\mathcal{P}, \mathcal{N})_n$ contains a semicycle of length longer than two, then there exists $(\mathcal{P}^*, \mathcal{N}^*)$ such that $|B_n(\mathcal{P})| \leq |B_n(\mathcal{P}^*)|$ and $(\mathcal{P}^*, \mathcal{N}^*)_n$ contains only semicycles of lengths one or two.

Proof. This again follows most easily from the path counting approach mentioned in the proof of Lemma 2. In particular, suppose that $\{i-1,i,i+1\} \subset \mathcal{P}$ and let $\mathcal{P}^* = \mathcal{P}/\{i\}$ and $\mathcal{N}^* = \mathcal{N} \cup \{i\}$. Set $Q$ and $Q^*$ to be the set of “paths” through $\mathcal{P}$ and $\mathcal{P}^*$, respectively, where $|B_n(\mathcal{P})| = ||Q||$ and $|B_n(\mathcal{P}^*)| = ||Q^*||$.

If $\pi = (p_1, p_2 \cdots, p_t) \in Q \cap (Q^*)^c$, then $p_j = i$ for some $j$, but for each such
\( \pi \), the path \( \pi^* = (p_1, p_2, \ldots, p_{j-1}, i-1, i, i+1, p_{j+1}, \ldots, p_t) \in Q^c \cap Q^* \). Thus, \( \|Q^*\| \geq \|Q\| \), and the proof is complete. \( \square \)

Here we will prove the following which limits our search space further.

**Lemma 4** If \((\mathcal{P}, \mathcal{N})_n\) contains two adjacent 2-cycles then \(|B_n(\mathcal{P})| \leq |B_n(\mathcal{P}^*)|\) for some \((\mathcal{P}^*, \mathcal{N}^*)\) such that each pair of 2-cycles in \((\mathcal{P}^*, \mathcal{N}^*)_n\) is separated by at least one 1-cycle.

**Proof.** First, assume that the partition has two adjacent 2-cycles, with the first of the two occurring at some \(4 \leq j \leq n-6\). Without loss of generality, we will assume that \(j \in \mathcal{P}\). The nine different possibilities are listed in Table 1. There we have used + and - to distinguish set membership (either \(\mathcal{P} (\mathcal{P}^*)\) or \(\mathcal{N} (\mathcal{N}^*)\)). An asterisk at the end of a sign configuration in Table 1 implies that all signs are reversed from that point onwards. That is, \(\mathcal{P}^* \cap \{j+6, \cdot \cdot \cdot, n\} = \mathcal{N} \cap \{j+6, \cdot \cdot \cdot, n\}\) and \(\mathcal{N}^* \cap \{j+6, \cdot \cdot \cdot, n\} = \mathcal{P} \cap \{j+6, \cdot \cdot \cdot, n\}\).

<table>
<thead>
<tr>
<th>Case</th>
<th>Segment of ((\mathcal{P}, \mathcal{N}))</th>
<th>Corresponding segment of ((\mathcal{P}^<em>, \mathcal{N}^</em>))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>++---+++---+++---</td>
<td>++---+++---+++---</td>
</tr>
<tr>
<td>2</td>
<td>++---+++---+++---</td>
<td>++---+++---+++---</td>
</tr>
<tr>
<td>3</td>
<td>++---+++---+++---</td>
<td>++---+++---+++---</td>
</tr>
<tr>
<td>4</td>
<td>++---+++---+++---</td>
<td>++---+++---+++---</td>
</tr>
<tr>
<td>5</td>
<td>++---+++---+++---</td>
<td>++---+++---+++---</td>
</tr>
<tr>
<td>6</td>
<td>++---+++---+++---</td>
<td>++---+++---+++---</td>
</tr>
<tr>
<td>7</td>
<td>--+++---+++---+++</td>
<td>--+++---+++---+++</td>
</tr>
<tr>
<td>8</td>
<td>--+++---+++---+++</td>
<td>--+++---+++---+++</td>
</tr>
<tr>
<td>9</td>
<td>--+++---+++---+++</td>
<td>--+++---+++---+++</td>
</tr>
</tbody>
</table>

Note that we have used Lemma 3 to eliminate any 3-cycles. Now, set \(t_i = |B_{j-4+i}(\mathcal{P})|\), for \(1 \leq i \leq 10\). Since the arguments are similar for each case, we will demonstrate the lemma for two of the cases.

(1) Case 1. By (6), we have \(t_5 = t_2 + t_3, t_6 = t_4 + t_5 = t_2 + t_3 + t_4, t_7 = t_2 + t_3 + t_4, t_8 = 2(t_2 + t_3 + t_4), t_9 = 2(t_2 + t_3 + t_4)\) and \(t_{10} = 4(t_2 + t_3 + t_4)\). Consider \((\mathcal{P}^*, \mathcal{N}^*)\) obtained by swapping the \(j + 1\) and \(j + 2\) elements between sets (see Table 1) and leaving the rest of \((\mathcal{P}, \mathcal{N})\) unchanged, and let \(s_i = |B_{j-4+i}(\mathcal{P}^*)|\), for \(1 \leq i \leq 10\). By (6), we have \(s_5 = s_1 + s_4, s_6 = s_2 + s_3 + s_5 = s_1 + s_2 + s_3 + s_4, s_7 = s_4 + s_6 = s_1 + s_2 + s_3 + 2s_4, s_8 = s_5 + s_7 = s_1 + s_2 + s_3 + 3s_4, s_9 = s_6 + s_8 = s_1 + s_2 + s_3 + 4s_4, s_{10} = s_7 + s_9 = s_1 + s_2 + s_3 + 6s_4\).
The results of these computations, upon noting that \( s_i = t_i \) for \( 1 \leq i \leq 4 \), are summarized in Table 2.

### Table 2

Table of computations for Case 1 in the proof of Lemma 4

<table>
<thead>
<tr>
<th>( i )</th>
<th>( s_i )</th>
<th>( t_i )</th>
<th>( t_i - s_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>( t_2 + t_3 + t_4 )</td>
<td>( t_1 + t_2 + t_3 + 2t_4 )</td>
<td>( t_1 + t_4 )</td>
</tr>
<tr>
<td>8</td>
<td>( 2(t_2 + t_3 + t_4) )</td>
<td>( 2t_1 + t_2 + t_3 + 3t_4 )</td>
<td>( 2t_1 - t_2 - t_3 + t_4 )</td>
</tr>
<tr>
<td>9</td>
<td>( 2(t_2 + t_3 + t_4) )</td>
<td>( 2t_1 + t_2 + t_3 + 3t_4 )</td>
<td>( 2t_1 - t_2 - t_3 + t_4 )</td>
</tr>
<tr>
<td>10</td>
<td>( 4(t_2 + t_3 + t_4) )</td>
<td>( 5t_1 + 3t_2 + 3t_3 + 7t_4 )</td>
<td>( 5t_1 - t_2 - t_3 + 3t_4 )</td>
</tr>
</tbody>
</table>

Now, note that by (6), \( t_4 \geq t_2 + t_3 \) and hence \( s_i \geq t_i \), for \( 7 \leq i \leq 10 \). Since \( P \cap \{ j + 3, j + 4, \cdots, n \} = P^* \cap \{ j + 3, j + 4, \cdots, n \} \), the result follows in this case.

### Table 3

Table of computations for Case 6 in the proof of Lemma 4

<table>
<thead>
<tr>
<th>( i )</th>
<th>( s_i )</th>
<th>( t_i )</th>
<th>( t_i - s_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>( t_3 + t_4 )</td>
<td>( t_1 + t_2 + t_3 + t_4 )</td>
<td>( t_1 + t_2 )</td>
</tr>
<tr>
<td>8</td>
<td>( t_2 + 2t_3 + 2t_4 )</td>
<td>( 2t_1 + 2t_2 + 2t_3 + 3t_4 )</td>
<td>( 2t_1 + t_2 + t_4 )</td>
</tr>
<tr>
<td>9</td>
<td>( t_3 + 3t_3 + 2t_4 )</td>
<td>( 3t_1 + 3t_2 + 2t_3 + 4t_4 )</td>
<td>( 3t_1 + 2t_2 - t_3 + 2t_4 )</td>
</tr>
<tr>
<td>10</td>
<td>( 2t_2 + 5t_3 + 4t_4 )</td>
<td>( 5t_1 + 5t_2 + 4t_3 + 6t_4 )</td>
<td>( 5t_1 + 3t_2 - t_3 + 2t_4 )</td>
</tr>
</tbody>
</table>

Now, note that by (6), \( t_4 \geq t_3 \) and hence \( s_i \geq t_i \), for \( 7 \leq i \leq 10 \). Since \( P \cap \{ j + 3, j + 4, \cdots, n \} = N^* \cap \{ j + 3, j + 4, \cdots, n \} \), the result follows in this case.

The remaining cases are proved similarly.

Now, assume that the partition has two adjacent 2-cycles, with the first of the two beginning at some \( j > n - 6 \). Using the result above, we have five cases to consider. The cases are listed in Table 4.
Table 4
Table of replacements for $j > n - 6$

<table>
<thead>
<tr>
<th>case</th>
<th>Segment of $(P, N)$</th>
<th>Corresponding segment of $(P^<em>, N^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$+-+-+$</td>
<td>$+-+-+$</td>
</tr>
<tr>
<td>2</td>
<td>$+-+-+-+---$</td>
<td>$+-+-+-+--$</td>
</tr>
<tr>
<td>3</td>
<td>$-+-+-+-+---$</td>
<td>$-+-+-+-+--$</td>
</tr>
<tr>
<td>4</td>
<td>$+-+-+-++--$</td>
<td>$+-+-+-++--$</td>
</tr>
<tr>
<td>5</td>
<td>$-+-+-+-+---$</td>
<td>$-+-+-+-+--$</td>
</tr>
</tbody>
</table>

We shall prove $|B_n(P^*)| \geq |B_n(P)|$ for Cases 1 and 5.

(1) Case 1. Here we have from (6),

$$|B_n(P)| \leq |B_{n-1}(P)| = |B_{n-1}(P^*)| \leq |B_n(P^*)|.$$  \hfill (9)

(2) Case 5. Following computations similar to those for when $4 \leq j \leq n - 6$, we have

$$|B_n(P)| = 4|B_{n-8}(P)| + 3|B_{n-7}(P)| + 5|B_{n-6}(P)| + |B_{n-5}(P)|$$
$$\leq 4|B_{n-8}(P)| + 3|B_{n-7}(P)| + 6|B_{n-5}(P)| + |B_{n-5}(P)|$$
$$= 4|B_{n-8}(P^*)| + 3|B_{n-7}(P^*)| + 6|B_{n-6}(P^*)| + |B_{n-5}(P^*)|$$
$$= |B_n(P^*)|,$$  \hfill (10)

where the second last equality in (10) follows since $B_{n-i}(P^*) = B_{n-i}(P)$ for $i \geq 5$.

The remaining cases follow similarly.

If $j < 4$, then we may apply the symmetry result in Lemma 2 to reduce to the case $j > n - 6$, and the result follows. \hfill $\square$

Employing Lemma 4, we also obtain the following.

**Lemma 5** If $(P, N)_n$ contains two 2-cycles separated by a single 1-cycle, then $|B_n(P)| \leq |B_n(P^*)|$ for some $(P^*, N^*)$ such that each pair of 2-cycles in $(P^*, N^*)_n$ is separated by at least two 1-cycles.

**Proof.** First, assume that the partition has two adjacent 2-cycles, separated by a single 1-cycle, with the first of the two beginning at some $4 \leq j \leq n - 7$. The four different possibilities are listed in Table 2. Again, we have used $+$ and $-$ to distinguish set membership (either $N$ or $P$), and an asterisk at the end of a sign configuration implies that all signs are reversed from that point.
onwards.

Table 5
Table of replacements for Lemma 5

<table>
<thead>
<tr>
<th>case</th>
<th>Segment of ((P,N))</th>
<th>Corresponding segment of ((P^<em>,N^</em>))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>++−−++−−++−−++</td>
<td>++−−++−−++−−++</td>
</tr>
<tr>
<td>2</td>
<td>++−−++−−++−−−</td>
<td>++−−++−−++−−*</td>
</tr>
<tr>
<td>3</td>
<td>−−−−++−−++−−++</td>
<td>−−−−++−−++−−*</td>
</tr>
<tr>
<td>4</td>
<td>−−−−++−−++−−−</td>
<td>−−−−−−−−−−−−*</td>
</tr>
</tbody>
</table>

Note that we have used Lemma 3 to eliminate any 3-cycles and Lemma 4 to eliminate adjacent 2-cycles. Now, set \(t_i = |B_{j-4+i}(P)|\) and \(s_i = |B_{j-4+i}(P^*)|\) for \(1 \leq i \leq 11\). Since the arguments are similar for each case, we will demonstrate the lemma for Case 1.

(1) Case 1. By (6), as in the proof of Lemma 4, upon noting that \(s_i = t_i\) for \(1 \leq i \leq 4\), we have the results in Table 6.

Table 6
Table of computations for Case 1 in the proof of Lemma 5

<table>
<thead>
<tr>
<th>(i)</th>
<th>(s_i)</th>
<th>(t_i)</th>
<th>(t_i - s_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>(t_2 + t_3 + t_4)</td>
<td>(2t_1 + 2t_2 + t_3 + 3t_4)</td>
<td>(2t_1 + t_2 + 2t_4)</td>
</tr>
<tr>
<td>9</td>
<td>(2t_2 + 4t_3 + 2t_4)</td>
<td>(3t_1 + 3t_2 + 2t_3 + 4t_4)</td>
<td>(3t_1 + t_2 - 2t_3 + 2t_4)</td>
</tr>
<tr>
<td>10</td>
<td>(3t_2 + 5t_3 + 3t_4)</td>
<td>(4t_1 + 4t_2 + 3t_3 + 6t_4)</td>
<td>(4t_1 + t_2 - 2t_3 + 3t_4)</td>
</tr>
<tr>
<td>11</td>
<td>(2t_2 + 4t_3 + 2t_4)</td>
<td>(4t_1 + 4t_2 + 3t_3 + 6t_4)</td>
<td>(4t_1 + 2t_2 - t_3 + 4t_4)</td>
</tr>
</tbody>
</table>

Now, note that by (6), \(t_4 \geq t_3\) and hence \(s_i \geq t_i\), for \(7 \leq i \leq 10\). Since \(P \cap \{j+3, j+4, \ldots, n\} = P^* \cap \{j+3, j+4, \ldots, n\}\), the result follows in this case.

The remaining cases are proved similarly.
Table 7  
Table of replacements for $j > n - 7$ in the proof of Lemma 4

<table>
<thead>
<tr>
<th>case</th>
<th>Segment of $(P, N)$</th>
<th>Corresponding segment of $(P^<em>, N^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$- + - - + +$</td>
<td>$- + - - + +$</td>
</tr>
<tr>
<td>2</td>
<td>$+ - - - - + - +$</td>
<td>$+ - - - - + - +$</td>
</tr>
<tr>
<td>3</td>
<td>$+ - - - - + - +$</td>
<td>$+ - - - + + - -$</td>
</tr>
</tbody>
</table>

We shall prove $|B_n(P^*)| \geq |B_n(P)|$ for Cases 1 and 2.

(1) Case 1. Here, as in (9), we have from (6), $|B_n(P)| \leq |B_{n-1}(P)| = |B_{n-1}(P^*)| \leq |B_n(P^*)|$.

(2) Case 2. Following computations similar to those in (10), we have

$$|B_n(P)| = 3|B_{n-9}(P)| + 2|B_{n-8}(P)| + 4|B_{n-7}(P)| + 2|B_{n-6}(P)| \leq 3|B_{n-9}(P^*)| + 2|B_{n-8}(P^*)| + 4|B_{n-7}(P^*)| + 2|B_{n-6}(P^*)| = |B_n(P^*)|. \quad (11)$$

The remaining cases follow similarly.

If $j < 4$, then we may apply Lemma 2 to reduce to the case $j > n - 7$, and the result follows. □

For fixed $n$, Lemmas 3, 4 and 5 allow us to consider only $(P, N)$ such that $(P, N)_n$ has semicycles of length at most 2, and any two 2-cycles of $(P, N)_n$ are separated by at least two 1-cycles.

We now turn to a proof of Theorem 1.

3 Proof of Theorem 1

In this section, we prove Theorem 1.

Proof of Theorem 1. Suppose that $\{b_i\}$ and $\{c_{i,j}\}$ satisfy (1) and define $P$ and $N$ as in the statement of Lemma 1. By Lemmas 1, 3, 4 and 5, we may suppose that for every $i$, $(P, N)_i$ consists of semicycles of length at most 2, such that any two 2-cycles are separated by at least two 1-cycles. Consideration of all such possible $(P, N)_n$, leads to the values in Table 8, for $U_i$, where $1 \leq i \leq 30$. Note that (4) holds for all $1 \leq n \leq 30$, hence suppose that the bound holds for all $n < N$ for some $N > 30$, and set $x = 3/2$.  

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Table 8

$U_i$ for $1 \leq i \leq 30$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$U_n$</th>
<th>$U_n^{1/n}$</th>
<th>$n$</th>
<th>$U_n$</th>
<th>$U_n^{1/n}$</th>
<th>$n$</th>
<th>$U_n$</th>
<th>$U_n^{1/n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000000</td>
<td>1.000000</td>
<td>11</td>
<td>54</td>
<td>1.437111</td>
<td>21</td>
<td>3022</td>
<td>1.464631</td>
</tr>
<tr>
<td>2</td>
<td>1.000000</td>
<td>1.000000</td>
<td>12</td>
<td>82</td>
<td>1.443725</td>
<td>22</td>
<td>4614</td>
<td>1.467402</td>
</tr>
<tr>
<td>3</td>
<td>1.259921</td>
<td>1.259921</td>
<td>13</td>
<td>128</td>
<td>1.452423</td>
<td>23</td>
<td>7202</td>
<td>1.471348</td>
</tr>
<tr>
<td>4</td>
<td>1.316074</td>
<td>1.316074</td>
<td>14</td>
<td>186</td>
<td>1.452473</td>
<td>24</td>
<td>10458</td>
<td>1.470541</td>
</tr>
<tr>
<td>5</td>
<td>1.379730</td>
<td>1.379730</td>
<td>15</td>
<td>275</td>
<td>1.454193</td>
<td>25</td>
<td>15470</td>
<td>1.470888</td>
</tr>
<tr>
<td>6</td>
<td>1.383088</td>
<td>1.383088</td>
<td>16</td>
<td>403</td>
<td>1.454895</td>
<td>26</td>
<td>22672</td>
<td>1.470682</td>
</tr>
<tr>
<td>7</td>
<td>1.408544</td>
<td>1.408544</td>
<td>17</td>
<td>615</td>
<td>1.458987</td>
<td>27</td>
<td>34616</td>
<td>1.472724</td>
</tr>
<tr>
<td>8</td>
<td>1.424971</td>
<td>1.424971</td>
<td>18</td>
<td>960</td>
<td>1.464474</td>
<td>28</td>
<td>54032</td>
<td>1.475786</td>
</tr>
<tr>
<td>9</td>
<td>1.429969</td>
<td>1.429969</td>
<td>19</td>
<td>1394</td>
<td>1.463819</td>
<td>29</td>
<td>78460</td>
<td>1.474962</td>
</tr>
<tr>
<td>10</td>
<td>1.434895</td>
<td>1.434895</td>
<td>20</td>
<td>2062</td>
<td>1.464584</td>
<td>30</td>
<td>116062</td>
<td>1.475105</td>
</tr>
</tbody>
</table>

Now, assume that $N \in P$ and consider the $N$-length sign configuration for $(P,N)_N$. Such a configuration must end in one of the 14 cases indicated in Figure 1. There, the set membership for $N$ is indicated at the top of the figure, followed by that for $N - 1$, etc., down the tree. Note that a triangle lies above each case number indicating the associated path.
For instance, Case 3 would cover all configurations ending in $- + + - +$ (i.e.,
$N, N - 2, N - 3 \in \mathcal{P}$ and $N - 1, N - 4 \in \mathcal{N}$) while Case 5 would cover all
those ending in $+ - - + + - - - - +$. We will prove the result for these
two cases. The details for the remaining cases are similar and are suggested
in Table 9.

(1) Case 3. ($- + + - +$) Here we have $N, N - 2, N - 3 \in \mathcal{P}$ and $N - 1, N - 4 \in$
\( N \). Form the induction hypothesis and (6), we have \(|B_{N-4}(P)| \leq x^{N-4},
|B_{N-3}(P)| \leq x^{N-3},
|B_{N-2}(P)| \leq x^{N-3}\) and \(|B_{N-1}(P)| \leq x^{N-1}\) and
\[
|B_N(P)| = |B_{N-1}(P)| + |B_{N-4}(P)|
\leq x^{N-1} + x^{N-4}
< x^N.
\tag{12}
\]
The final inequality follows since for \( f \) and \( g \) given by \( f(y) = y^4 \) and
\( g(y) = y^3 + 1, \) we have \( f(x) - g(x) > 0. \)

(2) Case 5. \((+++---+++---+)--)
Here we have \( N, N-3, N-5, N-7, N-10 \in \mathcal{P} \) and \( N-1, N-2, N-4, N-6, N-8, N-9 \in \mathcal{N}. \)
From the induction hypothesis and (6), we have
\[
|B_{N-10}(P)| \leq x^{N-10},
|B_{N-9}(P)| \leq x^{N-9},
|B_{N-8}(P)| \leq x^{N-9},
|B_{N-7}(P)| \leq x^{N-7},
|B_{N-6}(P)| = |B_{N-7}(P)| + |B_{N-10}(P)|
\leq x^{N-7} + x^{N-10},
|B_{N-5}(P)| = |B_{N-6}(P)| + |B_{N-8}(P)| + |B_{N-9}(P)|
\leq x^{N-7} + 2x^{N-9} + x^{N-10},
|B_{N-4}(P)| = |B_{N-5}(P)| + |B_{N-7}(P)|
\leq 2x^{N-7} + 2x^{N-9} + x^{N-10},
|B_{N-3}(P)| = |B_{N-4}(P)| + |B_{N-6}(P)|
\leq 3x^{N-7} + 2x^{N-9} + 2x^{N-10},
|B_{N-2}(P)| = |B_{N-3}(P)| + |B_{N-5}(P)|
\leq 4x^{N-7} + 4x^{N-9} + 3x^{N-10},
|B_{N-1}(P)| = |B_{N-3}(P)| + |B_{N-5}(P)|
\leq 4x^{N-7} + 4x^{N-9} + 3x^{N-10},
\tag{13}
\]
and hence
\[
|B_N(P)| = |B_{N-1}(P)| + |B_{N-2}(P)| + |B_{N-4}(P)|
\leq 10x^{N-7} + 10x^{N-9} + 7x^{N-10},
< x^N.
\tag{14}
\]
The final inequality in (14) follows since for \( f \) and \( g \) given by \( f(y) = y^{10} \)
and \( g(y) = 10y^3 + 10y + 7, \) we have \( f(x) - g(x) > 0. \)

The computations for the remaining cases are similar. The polynomials to
consider in each case are given in Table 9. For simplicity, the configuration for
each case is summarized via the number of adjacent 1-cycles between 2-cycles.
For instance the configuration for Case 8 is listed as 1531, as from Figure 1, this case covers \((P, N)\) ending in 

\[(+)-(+-++)--(+-)-(-).\]  

(15)

Table 9
Table of cases for Theorem 1

<table>
<thead>
<tr>
<th>case</th>
<th>config.</th>
<th>(g)</th>
<th>(f)</th>
<th>largest root of (f - g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>(x + x^3)</td>
<td>(x^4)</td>
<td>1.465571232</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>(1 + 2x + x^3)</td>
<td>(x^5)</td>
<td>1.486472477</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>(1 + x^3)</td>
<td>(x^4)</td>
<td>1.380277569</td>
</tr>
<tr>
<td>4</td>
<td>71</td>
<td>(5x + 3x^2 + 7x^3)</td>
<td>(x^9)</td>
<td>1.496605805</td>
</tr>
<tr>
<td>5</td>
<td>151</td>
<td>(7 + 10x + 10x^3)</td>
<td>(x^{10})</td>
<td>1.493615258</td>
</tr>
<tr>
<td>6</td>
<td>141</td>
<td>(5 + 6x + 7x^3)</td>
<td>(x^9)</td>
<td>1.495315552</td>
</tr>
<tr>
<td>7</td>
<td>731</td>
<td>(36x + 22x^2 + 53x^3)</td>
<td>(x^{14})</td>
<td>1.495687955</td>
</tr>
<tr>
<td>8</td>
<td>1531</td>
<td>(53 + 72x + 75x^3)</td>
<td>(x^{15})</td>
<td>1.493528881</td>
</tr>
<tr>
<td>9</td>
<td>1431</td>
<td>(36 + 44x + 53x^3)</td>
<td>(x^{14})</td>
<td>1.495124398</td>
</tr>
<tr>
<td>10</td>
<td>5331</td>
<td>(128x + 106x^2 + 164x^3)</td>
<td>(x^{17})</td>
<td>1.499868696</td>
</tr>
<tr>
<td>11</td>
<td>13331</td>
<td>(164 + 256x + 270x^3)</td>
<td>(x^{18})</td>
<td>1.498800208</td>
</tr>
<tr>
<td>12</td>
<td>12331</td>
<td>(128 + 159x + 164x^3)</td>
<td>(x^{17})</td>
<td>1.493134512</td>
</tr>
<tr>
<td>13</td>
<td>1231</td>
<td>(17 + 21x + 22x^3)</td>
<td>(x^{12})</td>
<td>1.491652213</td>
</tr>
<tr>
<td>14</td>
<td>121</td>
<td>(2 + 3x + 3x^3)</td>
<td>(x^8)</td>
<td>1.491661659</td>
</tr>
</tbody>
</table>

Note that the arguments for each case (and the induction hypothesis), would follow through as long as \(3/2\) was replaced with any constant larger than 1.499868696 and hence the assertion in the abstract that the order is \(strictly\) less than \((3/2)^n\).  \(\square\)

We are now in a position to prove Theorem 2.

**Proof of Theorem 2.** Considering \(b_n\) as a function of \(A = \{\alpha_{i,j}\}\) and \(b_0 = (b_{-3}, b_{-2}, b_{-1}, b_0)\) write \(b_n(A, b_0)\). Then

\[
b_n(A, b_0) = b_{-3}b_n(A, (1, 0, 0, 0)) + b_{-2}b_n(A, (0, 1, 0, 0)) + b_{-1}b_n(A, (0, 0, 1, 0)) + b_0b_n(A, (0, 0, 0, 1))
\]

(16)
Suppose that \((\mathcal{P}, \mathcal{N})\) is a partition of the set \(\{-3, -2, -1, 0, 1, 2, \cdots \}\) with \(\{-3, -2, -1, 0\} \subset \mathcal{P}\). Now, for given initial values \((b_{-3}, b_{-2}, b_{-1}, b_0) = (h_0, h_1, h_2, h_3)\), with \(h_i \in \{0, 1\}\) for \(0 \leq i \leq 3\), as in (6), define \(\{C_i\}_{i \geq -3}\), via
\[
C_i = h_{i+3} \quad \text{for} \quad i \in \{-3, -2, -1, 0\}
\]
and
\[
C_n = \begin{cases} 
- \sum_{i \in \{n-1,n-2,n-3,n-4\} \cap \mathcal{N}} C_i, & n \in \mathcal{P} \\
- \sum_{i \in \{n-1,n-2,n-3,n-4\} \cap \mathcal{P}} C_i, & n \in \mathcal{N}
\end{cases}
\]
for \(n \geq 1\).

As in Lemma 1, we have
\[
|h_i| \leq |C_i(\mathcal{P})|,
\]
for all \(i \geq -3\).

Table 10
Table of cases for the proof of Theorem 2

| Case | \((\mathcal{P}, \mathcal{N})\)_3 | \(|C_0|\) | \(|C_1|\) | \(|C_2|\) | \(|C_3|\) |
|------|------------------|--------|--------|--------|--------|
| 1    | ++--  | \(h_0\) | \(h_0 + h_1 + h_2 + h_3\) | \(h_1 + h_2 + h_3\) | \(h_2 + h_3\) |
| 2    | ++--  | \(h_0\) | \(h_0 + h_1 + h_2 + h_3\) | \(h_1 + h_2 + h_3\) | \(h_0 + 2h_1 + 2h_2 + 2h_3\) |
| 3    | ++--  | \(h_0\) | \(h_0 + h_1 + h_2 + h_3\) | \(h_0 + h_1 + h_2 + h_3\) | \(h_0 + h_1 + 2h_2 + 2h_3\) |
| 4    | ++--  | \(h_0\) | \(h_0 + h_1 + h_2 + h_3\) | \(h_0 + h_1 + h_2 + h_3\) | \(h_0 + h_1 + h_2 + h_3\) |
| 5    | ++--  | \(h_0\) | 0 | \(h_1 + h_2 + h_3\) | \(h_2 + h_3\) |
| 6    | ++--  | \(h_0\) | 0 | \(h_1 + h_2 + h_3\) | \(h_1 + h_2 + h_3\) |
| 7    | ++--  | \(h_0\) | 0 | 0 | \(h_2 + h_3\) |
| 8    | ++--  | \(h_0\) | 0 | 0 | 0 |

From Table 10, we see that for \((h_0, h_1, h_2, h_3) \in \{(1,0,0,0), (1,0,0,0), (1,0,0,0), (1,0,0,0)\}\) each value of \(|C_i|\) for \(0 \leq i \leq 3\) is maximized when \((h_0, h_1, h_2, h_3) = (0,0,0,1)\). Hence, from (16), and Theorem 1, we have
\[ |b_n(A, b_0)| \leq |b_{-3}| U_n + |b_{-2}| U_n + |b_{-1}| U_n + |b_0| U_n < (|b_{-3}| + |b_{-2}| + |b_{-1}| + |b_0|) (3/2)^n. \]  \hspace{1cm} (19)

\[ \square \]

4 Applications and Optimality of Theorem 1

In this section we discuss the optimality of Theorem 1 and its immediate application to bounding entries in inverses of 0, 1 banded matrices.

4.1 Optimality of Theorem 1

Let \((P, N)\) be the partition of \(\mathbb{Z}^+\) obtained via \(i \in P\) if and only if \(i \equiv 0 \text{ or } 3 \mod 5\). Then, it is not difficult to show by induction that for sufficiently large \(i\),

\[ B_i(P) = 8B_{i-5}(P) - 4B_{i-10}(P) + 2B_{i-15}(P). \]  \hspace{1cm} (20)

Solving the recurrence in (20), we have

\[ \lim_{i \to \infty} |B_i|^{1/i} \approx 1.496372348. \]  \hspace{1cm} (21)

Comparing the limit in (21) with the constant \(3/2\) (or the value \(1.499868696\); see the comment at the end of the proof of Theorem 1), the rate obtained in Theorem 1 is nearly optimal.

Computation of \(U_i\) for \(1 \leq i \leq 57\), suggests the optimal value of \(B_n\) is attained for \((P, N)\) by \(0, 3 \in P, 1, 2, 4 \in N\), and for \(n > 5\)

\[ j \in N \text{ if and only if } \begin{cases} j \equiv 1, 3, 4 \mod 5, & \text{if } n \equiv 0 \mod 5 \\ j \equiv 1, 3 \mod 5, & \text{if } n \equiv 1 \mod 5 \\ j \equiv 2, 4 \mod 5, & \text{if } n \equiv 2 \mod 5 \\ j \equiv 1, 2, 4 \mod 5, & \text{if } n \equiv 3 \mod 5 \\ j \equiv 1, 4 \mod 5, & \text{if } n \equiv 4 \mod 5 \end{cases}. \]  \hspace{1cm} (22)
4.2 An application to (0,1) banded matrices

Theorem 1 leads directly the following result.

**Corollary 1** Consider inverting the lower triangular matrix $L_n = [l_{i,j}]_{n \times n}$; ie. solving for $X_n = [x_{i,j}]_{n \times n}$ in the lower triangular linear system $L_nX_n = I_n$, where $I_n$ is the $n \times n$ identity matrix. Suppose that $l_{i,j} \in \{0,1\}$ for $1 \leq i \leq n$ and $1 \leq j \leq i$, with $l_{i,i} = 1$ for $1 \leq i \leq n$, and $l_{i,j} = 0$ for $i-j > 4$, then

$$|x_{k,s}| \leq U_{k-s} \tag{23}$$

for $1 \leq s \leq n$ and $s \leq k \leq n$, where $\{U_t\}$ is as in (4).

**Proof** This follows directly from the recurrence that arises when solving for the entries in each individual column of $X_n$. For further details and results for other classes of triangular matrices, see [3], [4] and [1]. □

For other results on bounding entries in inverses of (0,1) triangular matrices c.f. Graham and Sloane [10] and Marenich [13].

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**References**


