A CLASS OF SINGULAR NONLINEAR BOUNDARY VALUE PROBLEMS

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For Professor Lloyd Jackson
on the occasion of his 75th birthday

1 Introduction

The motivation for our work is the paper by Agarwal and O’Regan [1]. One of their theorems applies to the example

\[ y'' = -(y^{-\alpha} + 1)(1 - (-y')^\gamma), \quad 0 < t < 1, \]

\[ y'(0) = 0, \quad y(1) = 0, \quad \alpha > 0, \quad \gamma \geq 1, \]

(1)
(2)

to show that it has a solution \( y(t) \) where \( y(t) > 0 \) on \([0,1)\) for all \( \gamma \geq 1 \). In an attempt to gain further understanding of such examples, we consider a class of problems of the form

\[ y'' = g(t, y) f(y'), \]

\[ y'(0) = m \leq 0, \quad y(1) = 0. \]

(3)
(4)

We assume, as in the example, that our problem has a singularity at the right endpoint. Unlike most singular problems which have received attention recently, here \( y(t) \) and \( y'(t) \) appear multiplicatively in the same term, and the sign of \( y'' \) varies with the value of \( y' \). It is particularly significant in (1) that any positive function \( y(t) \) for which \( y'(t) \equiv -1 \) is a solution.

Our results for the general class (3), (4) applied to (1), (2) will

(i) show existence of a unique positive solution for which \( y'(t) > -1 \) for \( 0 \leq t < 1 \);

(ii) provide necessary and sufficient conditions for \( y'(1) > -1 \);
(iii) give asymptotic formulae for $y(t)$ and $y'(t)$ as $t \to 1^-$. 

In his paper [8], Taliaferro carries out a similar program for the problem

$$y'' = -\frac{\phi(t)}{y^\lambda}, \quad (5)$$

$$y(0) = y(1) = 0, \quad (6)$$

where $\lambda > 0$ and $\phi(t)$ is positive and continuous on $(0,1)$. Note that here $y'$ does not appear at all, although there is a singularity at both endpoints. Taliaferro shows existence of a unique positive solution to (5), (6) if and only if $\int_0^1 t(1-t)\phi(t) \, dt < \infty$. Analogously, we will show that a sufficient condition for existence of a positive solution of (3), (4) is $\int_0^1 (1-t)g(t,y) \, dt < \infty$, for each fixed $y > 0$. Taliaferro also gives a necessary and sufficient condition in order that $y'(t)$ be finite at an endpoint. For example, $y'(1)$ is finite if and only if

$$\int_{\frac{1}{2}}^1 \frac{\phi(t)}{(1-t)^\lambda} \, dt < \infty. \quad (7)$$

We obtain for our problem (3), (4) a similar condition for $\lim_{t\to1^-} y'(t) > -1$.

Taliaferro’s work on (5), (6) with respect to (i) has been extended since its original publication by Gatica, Oliker, and Waltman [5], Gatica, Hernandez, and Waltman [4], and by Baxley [2]. A survey of these results appears in [2]. Baxley [2, 3] also extended Taliaferro’s with respect to (ii), (iii). In [4], $y'$ appears linearly and later in [2, 3], $y'$ appears nonlinearly, but $y$ and $y'$ in these papers are additively separated.

Shin [7] proves existence of a positive solution for the example

$$y'' = \frac{\frac{1}{2}(1-t^2)y'}{y^2}, \quad 0 < t < 1,$$

$$y'(0) = -\frac{1}{2}, \quad y(1) = 0,$$

which is singular at $t = 1$ and includes $y$ and $y'$ in the same term of the differential equation.

Motivated by Shin’s example and other results in [1], Baxley and J. Martin (in work not yet published) focus on a family of nonlinear singular boundary value problems having the form

$$y'' = -\frac{\phi(t)|y'|^p}{y^\lambda}, \quad 0 < t < 1, \quad (8)$$

$$y'(0) = m < 0, \quad y(1) = 0, \quad (9)$$

where $\phi(t)$ is positive, continuous and bounded on $(0,1)$, $0 < p \leq 2$ and $\lambda > 0$. Common properties of (8), (9) and (3), (4) include the singularity at the right endpoint and the multiplicative appearance of $y$ and $y'$. The feature of (1) which sets it apart from (8) is
that while the right side of (8) is always negative for $y > 0$; the right side of (1) changes
sign at $y' = -1$. This sign change, which is characteristic of our class (3), leads to different
and interesting behavior of the solution. The main purpose of our work is to describe the
effect of this sign change.

All of our results will assume that the functions appearing in (3) and the number $m$ in
(4) satisfy the following hypotheses.

\( H_g : \ g(t, y) \in C^1([0, 1) \times (0, \infty)) \) and is positive and nonincreasing in $y$ for each fixed $t$.
   Moreover, \( \int_0^1 g(t, y)(1 - t) \, dt < \infty \), for each fixed $y > 0$.

\( H_f : \ f \in C^1[m_0, m_1] \), where $m_0 < 0$, $m \in (m_0, m_1)$ and
   \( f(m_0) = 0 \), \( f(z) < 0 \) for $m_0 < z < m_1$.

For the results in section 3, additional restrictions will be imposed.

The example of Agarwal and O’Regan discussed already is the case
\( g(t, y) = 1 + y^{-\alpha} \), \( f(z) = |z|^{\gamma} - 1 \), where $\gamma \geq 1$, $\alpha > 0$, $m_0 = -1$, $m = 0$, and $m_1 = 1$. It is trivial to verify
that the example satisfies the hypotheses above. Note that our hypotheses, while allowing
$g$ to be singular at $y = 0$ or $t = 1$, do not require a singularity.

In section 2, we prove existence of a unique positive solution to (3), (4). The results
in section 3 describe the behavior of the solution of the general problem (3), (4) for $t \approx 1$
in the case that $g$ is singular at $y = 0$. When applied to the example (1), (2), we have a
necessary and sufficient integral condition for \( \lim_{t \to 1^-} y'(t) > -1 \). The asymptotic formulae
for $g(t)$ and $y'(t)$ near $t = 1$ depend on whether or not \( \lim_{t \to 1^-} y'(t) > -1 \).

2 Existence and Uniqueness

Our goal in this section is to prove existence and uniqueness of a solution of the problem
(3), (4). Because of the possible singularity at $y = 0$, we first prove existence for the
boundary value problem consisting of (3) with the boundary conditions

\[
y'(0) = m \leq 0, \quad y(1) = y_1 > 0
\]

**Theorem 1** Suppose $H_f$ and $H_g$ are satisfied. Then (3), (10) has at least one positive
solution $y(t)$ which satisfies $m_0 < y'(t) \leq m$, $y(t) > y_1$ for $0 \leq t < 1$.

Proof: We temporarily remove the singularity at $y = 0$ by considering the modified problem,
consisting of (10) and the differential equation

\[
y'' = \hat{g}(t, y)\hat{f}(y'), \quad 0 < t < 1,\tag{11}
\]

where

\[
\hat{f}(z) = \begin{cases} f(m_1), & z \geq m_1 \\ f(z), & m_0 \leq z \leq m_1 \\ f(m_0), & z < m_0 \end{cases}, \tag{12}
\]

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and

\[ \hat{g}(t, y) = \begin{cases} g(t, y), & y \geq y_1 \\ g(t, y_1), & y < y_1 \end{cases}. \] \hspace{1cm} (13)

We shall use initial value methods to prove existence for the problem (11), (10). Let \( y(t) \) be the solution of (11) and the initial conditions

\[ y(0) = y_0, \quad y'(0) = m. \] \hspace{1cm} (14)

We first show that \( y(t) \) exists on the interval \([0, 1]\). Otherwise, \( t_0 = \sup \{ s \in (0, 1) : y(t) \) exists on \([0, s]) \} \) is less than 1. Since any linear function with slope \( m_0 \) is a solution of (11), then uniqueness for initial value problems and a concavity argument show immediately that \( m \geq y'(t) > m_0 \) for \( 0 \leq t < t_0 \). Therefore

\[ y(t) - y(0) = \int_0^t y'(s) \, ds \leq \int_0^t m \, ds = mt \]

on \( 0 \leq t < t_0 \). Hence, \( y(t) \leq y(0) + mt \leq y(0) \) on \( 0 \leq t < t_0 \). The sign assumptions on \( f \) and \( g \) imply \( y''(t) < 0 \) on \([0, t_0]\) and so \( y'(t) \) is decreasing. Thus \( y_2 = \lim_{t \to t_0^-} y'(t) \) exists.

Using \( y(t) - y(0) = \int_0^t y'(s) \, ds \), for \( 0 \leq t < t_0 \), we may take limits as \( t \to t_0^- \) and conclude that \( y_1 = \lim_{t \to t_0^-} y(t) \) exists. Thus, the solution \( y(t) \) may be continued to the right of \( t_0 \), contradicting the assumption that \([0, t_0) \) is the maximum interval of existence. Taking limits as before, \( y'(1) > m_0 \) and \( y(1) \) both exist. Let \( M = \max \{1, y_1\} \), with \( y_1 \) as in (10).

Define \( p > 0 \) by \( -p = \min \{ f(z) : m_0 \leq z \leq m_1 \} \). By \( H_g \) \( K = \int_0^1 g(t, M) \, dt < \infty \) and we may choose \( y_0 \) in (14) to satisfy \( y_0 > M - m + pK \). We shall now show that the solution \( y(t) \) of (11), (14) satisfies \( y(t) > M \) for \( 0 \leq t \leq 1 \). Otherwise, there exists \( t_0 \in (0, 1] \) for which \( y(t_0) = M \). Since \( y(t) > M \) on \( 0 \leq t < t_0 \), then \( H_g \) implies \( \hat{g}(t, y(t)) \leq \hat{g}(t, M) \).

By \( H_f \), \( f(y'(t)) < 0 \) and so

\[ y''(t) = \hat{g}(t, y(t)) \hat{f}(y'(t)) \geq \hat{g}(t, M) \hat{f}(y'(t)) \geq -p \hat{g}(t, M), \]

for \( 0 \leq t < t_0 \). Then

\[ y'(u) - y'(0) = \int_0^u y''(s) \, ds \geq -p \int_0^u \hat{g}(s, M) \, ds, \]

for \( 0 \leq u \leq t_0 \). Therefore,

\[ y'(u) \geq m - p \int_0^u \hat{g}(s, M) \, ds, \]

and thus

\[ y(t_0) - y(0) \geq mt_0 - p \int_0^{t_0} \int_0^u \hat{g}(s, M) \, ds \, du. \]
Replacing the upper limit \( t_0 \) of integration by 1 and interchanging the order of integration gives
\[
y(t_0) \geq y(0) + mt_0 - pK > M - m(1 - t_0) > M,
\]
a contradiction. Thus by choosing \( y_0 \) sufficiently large, the solution of the initial value problem (11), (14) overshoots the desired boundary condition at \( t = 1 \) Now choose \( y_0 \) in (14) as the value \( y_1 \) in (10). Then
\[
\int_0^1 m_0 \, ds \leq \int_0^1 y'(s) \, ds \leq \int_0^1 m \, ds.
\]
and so
\[
y_1 + m_0 t \leq y(t) \leq y_1 + m t < y_1, \text{ for } 0 \leq t < 1.
\]
Therefore, this solution undershoots the desired boundary condition at \( t = 1 \). Using continuous dependence in initial value problems, we conclude that (11), (10) has a solution \( y(t) \). Since this solution is concave down on \( 0 < t < 1 \), it follows that \( y(t) > y_1 \) on \([0, 1)\) and hence is a solution of the unmodified problem (3), (10).

We pass to our main existence theorem.

**Theorem 2** Suppose \( H_f \) and \( H_g \) are satisfied. Then the boundary value problem (3), (4) has at least one solution \( y(t) \) which satisfies \( m_0 < y'(t) \leq m, \ y(t) > 0 \) for \( 0 \leq t < 1 \).

Proof. By Theorem 1, the problem (3), (4) has a solution with \( y_1 = \frac{1}{n} \), for each \( n = 1, 2, 3, \cdots \) and \( m_0 \leq y_n'(t) \leq m \) for \( 0 \leq t < 1 \) for all \( n \). It follows that \( \{y_n\} \) is equicontinuous for \( 0 \leq t \leq 1 \). For \( 0 \leq t \leq 1 \), we have after integration from \( t \) to 1,
\[
-m_0(1 - t) + \frac{1}{n} \geq y_n(t) \geq \frac{1}{n}.
\]
It follows that
\[
\frac{1}{n} \leq y_n(t) \leq 1 - m_0, \text{ for } 0 \leq t \leq 1. \tag{15}
\]
Consequently \( \{y_n\} \) is uniformly bounded. Next, we show that \( \{y_n''\} \) is uniformly bounded on each closed subinterval \([0, t_1] \subset [0, 1]\). Since \( g(t, y) \) is nonincreasing for \( y > 0 \) then by (15)
\[
g(t, y_n(t)) \geq g(t, 1 - m_0).
\]
Let \( r = \min\{g(t, 1 - m_0) : 0 \leq t \leq \frac{1}{2}\} > 0 \). Then
\[
y_n''(t) = g(t, y_n(t))f(y_n'(t)) \leq rf(y_n'(t)), \ 0 \leq t \leq \frac{1}{2}.
\]
By \( H_f, -q = \max\{f(z) : z \in [\frac{m + m_0}{2}, m]\} < - \). Let
\[
R = \max\left\{\frac{m + m_0}{2}, -\frac{rq}{2}\right\} < 0.
\]

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We shall show that \( y_n'(\frac{1}{2}) \leq R \) for \( n=1, 2, 3, \ldots \). If \( y_n'(\frac{1}{2}) \leq \frac{m+m_0}{2} \) then we are done. Otherwise, \( m \geq y_n'(t) > \frac{m+m_0}{2} \) for \( 0 \leq t \leq \frac{1}{2} \). Thus

\[
y_n''(t) \leq rf(y_n'(t)) \leq -rq, \quad 0 \leq t \leq \frac{1}{2},
\]

and integrating gives

\[
y_n'(\frac{1}{2}) \leq m - \frac{rq}{2} \leq \frac{-rq}{2}.
\]

It follows that

\[
y_n'(t) \leq y_n'(\frac{1}{2}) \leq R, \quad \frac{1}{2} \leq t < 1.
\]

Choose \( t_1 \in \left(\frac{1}{2}, 1\right) \). Then integration gives

\[
y_n(1) - y_n(t_1) = \int_{t_1}^{1} y_n'(s) \, ds \leq R(1 - t_1)
\]

and since \( y_n(t) \) is decreasing,

\[
y_n(t) \geq y_n(t_1) > -R(1 - t_1) > 0,
\]

for \( 0 \leq t \leq t_1 \). Thus

\[
g(t, y_n(t)) \leq g(t, -R(1 - t_1)), \quad \text{for } 0 \leq t \leq t_1.
\]

Let \( M_1 = \max\{g(t, -R(1 - t_1)) : 0 \leq t \leq t_1\} \) and \( w = \min\{f(z) : z \in [m_0, m]\} \). Then

\[
y_n''(t) = g(t, y_n(t))f(y_n'(t)) \geq M_1 f(y_n'(t)) \geq M_1 w, \quad \text{for } 0 \leq t \leq t_1.
\]

Hence \( \{y_n''(t)\} \) is uniformly bounded on \([0, t_1]\) where \( t_1 \in [\frac{1}{2}, 1) \). Thus \( \{y_n''\} \) is equicontinuous on \([0, t_1]\). We may apply Ascoli’s theorem to get a subsequence of \( \{y_n\} \) which converges uniformly on \([0, 1]\) and then use Ascoli’s theorem with a diagonalization argument to get a further subsequence \( \{u_n\} \) of \( \{y_n\} \) so that \( \{u_n\} \) converges uniformly on \([0, 1]\) and \( \{u_n'\} \) converges uniformly on each closed subinterval of \([0, 1]\). Letting \( y(t) = \lim_{n \to \infty} u_n(t) \), it is easy to see that \( \{y_n'(t)\} \) converges to \( y'(t) \) on \([0, 1]\). Further from (16) we have \( y(t) \geq -R(1 - t_1) > 0 \) for \( 0 \leq t \leq t_1 \). We next use the Lebesgue bounded convergence theorem to show that \( \{y_n''(t)\} \) converges to \( y''(t) \) on \([0, 1]\) and is thus a solution of (3). For each \( n \geq 1 \)

\[
u_n''(t) = g(t, y_n(t))f(u_n'(t))
\]

and \( u_n'(0) = m \). For \( t_1 \in (\frac{1}{2}, 1) \), it follows from (16) that we may let \( n \to \infty \) in (18) to get

\[
w(t) \equiv \lim_{n \to \infty} u_n''(t) = g(t, y)f(y'(t)).
\]
Let $P$ be the maximum of $|f(z)|$ for $m_0 \leq z \leq m_1$. Using (17) with $t_1 = \max\{t, \frac{3}{4}\}$, we have
\[ |u''_n(s)| = |g(s, u_n(s)) f(u'_n(s))| \leq M_1 P, \quad 0 \leq s \leq t. \]

Since
\[ u'_n(t) - u'_n(0) = \int_0^t u''_n(s) \, ds, \quad \text{for } 0 \leq t < 1, \quad \text{(19)} \]
we may use the Lebesgue bounded convergence theorem to let $n \to \infty$ in (19) and get
\[ \lim_{n \to \infty} u'_n(t) = \lim_{n \to \infty} u'_n(0) + \int_0^t \lim_{n \to \infty} u''_n(s) \, ds, \]
which implies
\[ y'(t) = y'(0) + \int_0^t w(s) \, ds, \quad \text{for } 0 \leq t < 1. \]

Taking the derivative of both sides yields
\[ y''(t) = w(t). \]

Therefore $y(t)$ is a solution to (3). As before, uniqueness implies $y'(t) > m_0$ for $0 \leq t < 1$. Since $u'_n(0) = m$ for all $n \geq 1$, then $y'(0) = \lim_{n \to \infty} u'_n(0) = m$. Also $y_n(1) = \frac{1}{n}$ implies $\lim_{n \to \infty} y_n(1) = 0$, so $y(1) = \lim_{n \to \infty} u_n(1) = 0$. Thus $y(t)$ satisfies (4).

We now come to our uniqueness result.

**Theorem 3** Suppose $H_f$ and $H_g$ are satisfied. Then the boundary value problem (3) and (4) has at most one positive solution.

Proof. Suppose $y_1(t)$ and $y_2(t)$ satisfy (3), (4) and $y_1(t) \neq y_2(t)$. Without loss of generality, there exists a $t_1 \in [0, 1)$ such that $y_1(t_1) > y_2(t_1)$. Let $u(t) = y_1(t) - y_2(t)$; then $u(t_1) > 0$. For convenience we let $F(t, y, y') = g(t, y) f(y')$. Then
\[ u'' = y''_1 - y''_2 = F(t, y_1, y'_1) - F(t, y_2, y'_2) \\
= \left[ F(t, y_1, y'_1) - F(t, y_1, y'_2) \right] + \left[ F(t, y_1, y'_2) - F(t, y_2, y'_2) \right] \]

Let I be any closed subinterval of $[0, 1)$ so that $y_1(t) > y_2(t)$ on I. Then we have
\[ u''(t) + G(t) u'(t) + H(t) u(t) = 0, \quad \text{for } t \in I, \quad \text{(20)} \]

where
\[ G(t) = \begin{cases} - \left[ \frac{F(t, y_1(t), y'_1(t)) - F(t, y_1(t), y'_2(t))}{y'_1(t) - y'_2(t)} \right], & \text{if } y'_1(t) \neq y'_2(t) \\ 0, & \text{if } y'_1(t) = y'_2(t) \end{cases} \quad \text{(21)} \]

and
\[ H(t) = - \left[ \frac{F(t, y_1(t), y'_2(t)) - F(t, y_2(t), y'_2(t))}{y_1(t) - y_2(t)} \right]. \quad \text{(22)} \]

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Let \( a \in (0, t_1) \). Then on the interval \([a, 1]\), \( u(t) \) has a positive maximum at some \( c \in [a, 1] \); \( c \neq 1 \) since \( u(1) = 0 \). Thus \( u(c) \geq u(t) \) for every \( t \in [a, 1] \). Since \( u(1) = 0 \), we may choose \( b \in (c, 1) \) such that \( u(b) = \frac{u(c)}{2} \) and \( u(t) \geq \frac{u(c)}{2} \) for \( c \leq t \leq b \). Then on the interval \([c, b]\), \( u(t) \) has a positive maximum at \( c \) and \( y_1(t) > y_2(t) \) on \([c, b]\). By the mean value theorem

\[
G(t) = -g(t, y_1(t))f'(z(t)), \quad \text{for } c \leq t \leq b,
\]

(23)

where \( z(t) \) is between \( y'_1(t) \) and \( y'_2(t) \). Since \( y''_1(t) < 0 \) and \( y''_2(t) < 0 \) then,

\[
\begin{align*}
y'_1(b) & \leq y'_1(t) \leq y'_1(c) < m \\
y'_2(b) & \leq y'_2(t) \leq y'_2(c) < m
\end{align*}
\]

Let

\[
L = \max\{y'_1(c), y'_2(c)\} < m,
\]

\[
S = \min\{y'_1(b), y'_2(b)\} > m_0.
\]

Since \( f'(z) \) is continuous on \([S, L]\), then \( f'(z) \) is bounded on \([S, L]\). Noting that \( g(t, y_1(t)) \) is continuous on \([c, b]\); it follows that \( G(t) \) is bounded on the closed interval \([c, b]\).

Note that

\[
H(t) = -\frac{f(y'_2(t))}{y_1(t) - y_2(t)} \left[ g(t, y_1(t)) - g(t, y_2(t)) \right].
\]

Since \( H(t) \) is the product of continuous functions on \([c, b]\), then \( H(t) \) is bounded there. Clearly, \( H(t) \leq 0 \).

Hence \( G(t) + (t - c) \) \( H(t) \) is bounded on \([c, b]\) and by the maximum principle [6], \( u'(c) < 0 \). It follows that \( c = a \) and \( u'(a) < 0 \). Hence \( u'(t) < 0 \) on \((0, t_1)\) and the maximum of \( u(t) \) on \((0, t_1)\) occurs at \( 0 \) with \( u(0) > 0 \).

We next use the maximum principle to show that \( u'(0) < 0 \) which contradicts the fact that \( u'(0) = y'_1(0) - y'_2(0) = m - m = 0 \). The verification that \( G(t) \) and \( H(t) \) are bounded on any closed subinterval of \((0, t_1)\) and that \( H(t) \leq 0 \) on \((0, t_1)\) is the same as before. Since \( f \in C^1[m_0, m_1] \) then \( f' \) is bounded on \([m_0, m_1]\). Thus by (23) \((c = 0)\), \( G(t) \) is bounded above near \( 0 \). Hence by the maximum principle, \( u'(0) < 0 \) giving our contradiction and completing the proof of uniqueness.

### 3 Qualitative Behavior

Under the hypothesis of Theorem 3, we now know that (3), (4) has a unique positive solution \( y(t) \). Moreover, \( y'(t) \) is decreasing, so \( \lim_{t \to 1^-} y'(t) = -\beta \geq m_0 \) exists. Whether or not \( \lim_{t \to 1^-} y'(t) - \beta > m_0 \) depends on the finiteness of \( \int_0^1 g(s, k(1 - s)) \, ds \), for certain values of \( k > 0 \).
Theorem 4 Suppose $H_f$ and $H_g$ are satisfied. Then a necessary and sufficient condition for $\lim_{t \to 1^-} y'(t) > m_0$ is that $\int_0^1 g(s, k(1-s)) \, ds < \infty$ for some $k \in (0, -m_0)$.

Proof. Suppose $\lim_{t \to 1^-} y'(t) = -\beta > m_0$. Since $y'(t)$ is decreasing on $[0, 1]$, then $y'(t) > -\beta$ for $0 \leq t < 1$. Thus
\[ y(1) - y(t) > \int_t^1 -\beta \, ds = -\beta(1-t), \]
and so
\[ y(t) < \beta(1-t) \equiv w(t). \]

It follows that
\[ y''(t) = g(t, y(t))f(y'(t)) \leq g(t, w(t))f(y'(t)) \]

since $g$ is nonincreasing in $y$ and $f(y')$ is negative. By $H_f$, $-q \equiv \max\{f(z) : -\beta \leq z \leq m\} < 0$ and so for $0 < t < 1$, we have
\[ -\beta - m < y'(t) - y'(0) = \int_0^t y''(s) \, ds \leq -q \int_0^t g(s, w(s)) \, ds. \]

Negating this inequality and letting $t \to 1^-$, we obtain
\[ \int_0^1 g(s, w(s)) \, ds \leq \frac{\beta + m}{q} < \infty. \]

Note that $k \equiv \beta \in (0, -m_0)$.

We prove the converse by contradiction. Suppose that $\lim_{t \to 1^-} y'(t) = m_0$. First we construct a sequence $\{t_n\} \subset (0, 1)$ which converges to 1 and has the property that $f(y'(t_n)) \geq f(y'(t_n))$ for $t_n \leq s \leq 1$. Choose $w_1 \in (m_0, m)$. Since $f$ is continuous on $[m_0, w_1]$, there exists $z_1 \in (m_0, w_1]$ so that $f(z_1) \leq f(z)$ for $m_0 \leq z \leq w_1$. Then choose $w_2$ as the midpoint of $[m_0, z_1]$ and $z_2 \in (m_0, w_2]$ so that $f(z_2) \leq f(z)$ for $m_0 \leq z \leq w_2$.

Continuing by induction, we choose $w_n$ as the midpoint of $[m_0, z_{n-1}]$ and $z_n \in (m_0, w_n]$ so that $f(z_n) \leq f(z)$ for $m_0 \leq z \leq w_n$. Then $\{z_n\}$ is decreasing and $z_n \to m_0$ as $n \to \infty$. Since $y'(t)$ decreases from $m$ to $m_0$ as $t$ increases from 0 to 1, then for each $z_n$ there exists a unique $t_n$ so that $y'(t_n) = z_n$. Clearly $\{t_n\}$ is increasing and $t_n \to 1$ as $n \to \infty$. Our construction guarantees that $f(y'(s)) \geq f(y'(t_n))$ for $t_n \leq s \leq 1$. With this preparation, for $t_n < u < 1$, we have
\[ y'(u) - y'(t_n) = \int_{t_n}^u y''(s) \, ds \geq f(y'(t_n)) \int_{t_n}^u g(s, y(s)) \, ds. \quad (24) \]

We complete the proof by showing that $\int_0^1 g(s, k(1-s)) \, ds = \infty$ for every $k \in (0, -m_0)$.
For any such $k$, there exists $\delta > 0$ so that $y'(s) < -k$ for $1 - \delta < s < 1$. Therefore for $1 - \delta < t < 1$
\[ y(1) - y(t) = \int_t^1 y'(s) \, ds \leq -k(1-t) \]

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and so \( y(t) \geq k(1 - t) \). Now we choose \( N \) so that \( n \geq N \) implies \( t_n > 1 - \delta \). Using (24) and the fact that \( g(t, y) \) is nonincreasing in \( y \), we have

\[
\frac{y'(u) - y'(t_n)}{f(y'(t_n))} \leq \int_{t_n}^{u} g(s, k(1 - s)) \, ds.
\]

Now suppose \( \int_0^1 g(s, k(1 - s)) \, ds < \infty \). Letting \( u \to 1^- \) we obtain

\[
\frac{y'(1) - y'(t_n)}{f(y'(t_n))} \leq \int_{t_n}^{1} g(s, k(1 - s)) \, ds.
\]  
(25)

Since

\[
\lim_{n \to \infty} \frac{f(y'(t_n))}{y'(1) - y'(t_n)} = \lim_{z \to m_0^+} \frac{f(z)}{m_0 - z} = -f'(m_0) > 0,
\]

the left side of (25) tends to a positive finite limit as \( n \to \infty \). However, the right side of (25) tends to 0, giving a contradiction.

**Corollary 1** Suppose the hypotheses of Theorem 4 are satisfied and \( g \) has the form

\[
g(t, y) = \frac{\phi(t)}{h(y)}.
\]

where \( h(y) \sim y^\alpha \) as \( y \to 0^+ \), for some \( \alpha > 0 \). Then \( \lim_{t \to 1^-} y'(t) > m_0 \) if and only if \( \int_0^1 \frac{\phi(t)}{(1 - t)^\alpha} \, dt < \infty \).

**Proof:** By hypothesis, for \( 0 < \epsilon < 1 \), there exists \( y_0 \in (0, 1) \) such that \( 1 - \epsilon < \frac{y^\alpha}{h(y)} < 1 + \epsilon \) for \( 0 < y < y_0 \). For any fixed \( k > 0 \), there exists \( t_0 \in (0, 1) \) so that \( 0 < k(1 - t) < y_0 \) for \( t_0 < t < 1 \). From

\[
\int_{t_0}^{1} g(s, k(1 - s)) \, ds = \int_{t_0}^{1} \frac{\phi(s)}{h(k(1 - s))} \, ds = \frac{1}{k^\alpha} \int_{t_0}^{1} \frac{k(1 - s)}{h(k(1 - s))} \, ds,
\]

we have

\[
\frac{1}{k^\alpha} (1 - \epsilon) \int_{t_0}^{1} \frac{\phi(s)}{(1 - s)^\alpha} \, ds < \int_{t_0}^{1} g(s, k(1 - s)) \, ds < \frac{1}{k^\alpha} (1 + \epsilon) \int_{t_0}^{1} \frac{\phi(s)}{(1 - s)^\alpha} \, ds.
\]

Thus \( \int_{t_0}^{1} g(s, k(1 - s)) \, ds < \infty \) for some \( k > 0 \) if and only if \( \int_{t_0}^{1} \frac{\phi(s)}{(1 - s)^\alpha} \, ds < \infty \).

Since \( \phi(t) \) can be replaced by a positive multiple of itself without affecting the truth of this corollary, the corollary is true if \( h(y) \sim cy^\alpha \) for any positive constant \( c \). This corollary
applies to the example (1), (2) of Agarwal and O’Regan to show that \( \lim_{t \to 1^-} y'(t) > -1 \) if and only if \( 0 < \alpha < 1 \); for \( \alpha \geq 1 \), the slope of the solution tends to \(-1\) as \( t \to 1^- \).

We now seek more information regarding the asymptotic behavior of \( y(t) \) and \( y'(t) \) as \( t \to 1^- \). For this purpose, we continue with the scenario or the above corollary and require knowledge of the asymptotic behavior of \( \phi(t) \), so we assume henceforth that

\[
\phi(t) \sim c(1-t)^\lambda, \quad c > 0, \quad \text{as } t \to 1^-.
\]  

We first examine the situation when \( \int_0^1 \frac{\phi(t)}{(1-t)^\alpha} \, dt < \infty \). It is easy to see that this condition is equivalent to \( \lambda - \alpha + 1 > 0 \).

**Theorem 5** Suppose the hypotheses of Corollary 1 and (26) are satisfied and \( \lambda - \alpha + 1 > 0 \). Then

\[
y'(t) + \beta \sim -\frac{cf(-\beta)}{\beta^\alpha} \frac{(1-t)^{\lambda-\alpha+1}}{(\lambda - \alpha + 1)}, \quad \text{as } t \to 1^- \tag{27}
\]

and

\[
y(t) - \beta(1-t) \sim \frac{cf(-\beta)}{\beta^\alpha} \frac{(1-t)^{\lambda-\alpha+2}}{(\lambda - \alpha + 1)(\lambda - \alpha + 2)}, \quad \text{as } t \to 1^- \tag{28}
\]

Proof. Using L’hopital’s rule we obtain

\[
\lim_{t \to 1^-} \frac{y'(t) + \beta}{(1-t)^{\lambda-\alpha+1}} = \lim_{t \to 1^-} \frac{y''(t)}{-h(y(t))(\lambda - \alpha + 1)(1-t)^{\lambda-\alpha}}
\]

\[
= \lim_{t \to 1^-} \frac{\phi(t) f(y(t))}{(1-t)^{\lambda-\alpha+1}}\frac{f(y(t))}{y(t)} \lim_{t \to 1^-} \left( \frac{1-t}{y(t)} \right)^\alpha
\]

\[
= \frac{-cf(-\beta)}{\beta^\alpha(\lambda - \alpha + 1)},
\]

which implies (27). Applying L’hopital’s rule to

\[
\lim_{t \to 1^-} \frac{y(t) - \beta(1-t)}{(1-t)^{\lambda-\alpha+2}}\left( \frac{1-t}{y(t)} \right)^\alpha
\]

leads immediately to (28).

Now we consider the case where \( \int_0^1 \frac{\phi(t)}{(1-t)^\alpha} \, ds = \infty \) in which case \( \lambda - \alpha + 1 \leq 0 \).
Lemma 1 Suppose the hypotheses of Corollary 1 and (26) are satisfied and \( \lambda - \alpha + 1 \leq 0 \). Then

\[
L = \lim_{t \to 1^-} \frac{(1 - t)^{\alpha - \lambda}(y'(t) - m_0)}{m_0(t - 1) - y(t)}
\]

exists where

\[
L = 1 - c \frac{f'(m_0)}{(-m_0)^\alpha}, \quad \text{if } \lambda - \alpha + 1 = 0 \tag{29}
\]

\[
L = -c \frac{f'(m_0)}{(-m_0)^\alpha}, \quad \text{if } \lambda - \alpha + 1 < 0 \tag{30}
\]

Proof. Using L'Hopital's rule gives

\[
\lim_{t \to 1^-} \frac{(1 - t)^{\alpha - \lambda}(y'(t) - m_0)}{m_0(t - 1) - y(t)} = \lim_{t \to 1^-} \frac{(1 - t)^{\alpha - \lambda}(y'(t)) - (y'(t) - m_0)(\alpha - \lambda)(1 - t)^{\alpha - \lambda - 1}}{m_0 - y'(t)}
= \lim_{t \to 1^-} \frac{(1 - t)^\alpha}{(y(t))^\alpha} \frac{f(y'(t))}{m_0 - y'(t)} + \lim_{t \to 1^-} (\alpha - \lambda)(1 - t)^{\alpha - \lambda - 1}.
\]

Suppose \( \lambda - \alpha + 1 = 0 \). Then

\[
\lim_{t \to 1^-} \frac{(1 - t)^{\alpha - \lambda}(y'(t) - m_0)}{m_0(t - 1) - y(t)} = c \left(\frac{1}{m_0}\right)^\alpha \lim_{z \to m_0} \left[\frac{-(f(z) - f(m_0))}{z - m_0}\right] + 1
= c \left(\frac{1}{m_0}\right)^\alpha (1 - f'(m_0)) + 1,
\]

leading immediately to (29). If \( \lambda - \alpha + 1 < 0 \), the limit is the same except that the last term is 0 instead of 1.

When \( \lambda - \alpha + 1 \leq 0 \), we have been unable to obtain formulae which directly describe the asymptotic behavior of \( y(t) \) or \( y'(t) \). Therefore, we instead consider the behavior of \( \ln |m_0(t - 1) - y(t)| \) and \( \ln |y'(t) - m_0| \). By studying how fast these functions approach \(-\infty\), we gain some knowledge about how quickly \( m_0(t - 1) - y(t) \) and \( y'(t) - m_0 \) approach 0.

Theorem 6 Suppose the hypotheses of Corollary 1 and (26) are satisfied. If \( \lambda - \alpha + 1 = 0 \), then

\[
\ln |m_0(t - 1) - y(t)| \sim \left[1 - c \frac{f'(m_0)}{(-m_0)^\alpha}\right] \ln |1 - t|, \text{ as } t \to 1^-,
\tag{31}
\]

and if \( f'(m_0) \neq 0 \)

\[
\ln |y'(t) - m_0| \sim -c \frac{f'(m_0)}{(-m_0)^\alpha} \ln |1 - t|, \text{ as } t \to 1^-.
\tag{32}
\]
If \( \lambda - \alpha + 1 < 0 \) and \( f'(m_0) \neq 0 \) then

\[
\ln |m_0(t-1) - y(t)| \sim -\frac{c f'(m_0)}{(-m_0)^{\alpha/2} (\lambda - \alpha + 1)} (1-t)^{\lambda-\alpha+1}, \quad \text{as } t \to 1^-,
\]

(33)

and

\[
\ln |y'(t) - m_0| \sim -\frac{c f'(m_0) (1-t)^{\lambda-\alpha+1}}{(-m_0)^{\alpha/2} (\lambda + 1 - \alpha)}, \quad \text{as } t \to 1^-
\]

(34)

Proof. We begin with (31). Using L’Hopital’s rule and (29) in Lemma 1, we get

\[
\lim_{t \to 1^-} \frac{\ln |m_0(t-1) - y(t)|}{\ln |1-t|} = \lim_{t \to 1^-} \frac{(1-t) (y'(t) - m_0)}{m_0(t-1) - y(t)}
\]

\[
= 1 - cf'(m_0) \left( \frac{-1}{m_0} \right)^\alpha,
\]

which implies (31). We next prove (33). Using (30) in Lemma 1, we now have

\[
\lim_{t \to 1^-} \frac{\ln |m_0(t-1) - y(t)|}{-(1-t)^{\lambda+1-\alpha}} = \frac{-1}{(\lambda + 1 - \alpha)} \lim_{t \to 1^-} \frac{1}{m_0(t-1) - y(t)} \frac{(1-t)^{\alpha-\lambda} (y'(t) - m_0)}{cf'(m_0)}
\]

\[
= \frac{1}{(\lambda + 1 - \alpha)} \left( \frac{-1}{m_0} \right)^\alpha,
\]

and hence (33).

To prove (32) and (34) we first note that it follows easily from L’Hopital’s rule that

\[
\int_0^t \frac{\phi(s)}{(1-s)^{\alpha}} ds \sim \int_0^t c(1-s)^{\lambda-\alpha} ds, \quad \text{as } t \to 1^-.
\]

Thus

\[
\lim_{t \to 1^-} \frac{\ln |y'(t) - m_0|}{\int_0^t \frac{\phi(s)}{(1-s)^{\alpha}} ds} = \lim_{t \to 1^-} \frac{\ln |y'(t) - m_0|}{\int_0^t c(1-s)^{\lambda-\alpha}} ds
\]

\[
= \lim_{t \to 1^-} \frac{(1-t)^{\alpha/2}}{y(t)^{\alpha/2}} \frac{f(y'(t))}{y'(t) - m_0}
\]

\[
= \frac{f'(m_0)}{(-m_0)^{\alpha/2}}.
\]

Evaluating the integral in the denominator in the two cases \( \lambda - \alpha + 1 = 0 \) and \( \lambda - \alpha + 1 < 0 \) gives the results (32) and (34).

It is easy to apply Theorems 5 and 6 to the example (1), (2) of Agarwal and O’Regan to determine the behavior of the solution near \( t = 1 \).
References


