SOME NEW RESULTS ON DIFFERENCE EQUATIONS OF
CONVOLUTION TYPE

By

NATHANIEL G. VISH

A Thesis Submitted to the Graduate Faculty of
WAKE FOREST UNIVERSITY
in Partial Fulfillment of the Requirements
for the Degree of
MASTER OF ARTS
in the Department of Mathematics
May 2009
Winston-Salem, North Carolina

Approved By:

Kenneth S. Berenhaut, Ph.D., Advisor

Examing Committee:

Stephen B. Robinson, Ph.D., Chairperson
Jennifer B. Erway, Ph.D.
Table of Contents

Acknowledgments ............................................................... iii
Abstract ................................................................................ iv
Chapter 1 Introduction.......................................................... 1
Chapter 2 Equations of convolution type with monotone coefficients 5
Chapter 3 A 1-norm Bound for Inverses of Triangular Matrices with
Monotone Entries ................................................................. 17
Chapter 4 An Optimal Bound for Inverses of Triangular Matrices
with Monotone Entries ........................................................... 28
Chapter 5 Conclusion and Future Research ......................... 35
Bibliography ........................................................................... 36
Vita......................................................................................... 1
Acknowledgments

I would like to thank Dr. Kenneth S. Berenhaut, a stalwart advisor and friend, for his help during my time at Wake Forest. He has taught me a great deal about mathematics and compassion, and I owe much of my success in this program to his guidance.

Dr. Jennifer Erway has been kind enough to serve on my thesis review committee, and provided me with a wealth of opportunities to hone my classroom teaching skills while working as a TA under her guidance. This experience has been invaluable, and I know will serve me well as I move beyond Wake Forest.

I am grateful to Dr. Stephen Robinson for chairing my committee, as well as for the exceptional efforts he has made to assist the math graduate students on a day-to-day basis. His work to ease the strain of increasing university fees and his gift of the much-loved graduate meal card are greatly appreciated.

My officemates Christy, Jonathan, Justin and Tommy have my gratitude for many good times, too much information, late-night theology discussions, obnoxious ringtones, and lots of shared peanut butter and jelly. I thank Tommy especially for his work on papers contained in this thesis.

I extend special thanks to my parents Theodore and Denise, and my sisters Kathryn and Esther. They have been constant in my struggles, partners in my joy, and always ready to lend a listening ear. I cannot thank them enough for bearing with me through the trials of this journey.

Finally, I would like to thank Joanna Cutrara. Her love, prayers, and support have brought me through my last semester at Wake with far more grace than I could have mustered on my own. She has been a source of clarity when I needed it most, and an unfailing encouragement.
Abstract

In this thesis we consider convolution type linear difference equations with coefficients satisfying some monotonicity properties. Methods from renewal theory are employed to obtain easily verified conditions for asymptotic stability of the zero solution, in terms of the coefficient sequence. Explicit bounds and rates of convergence are considered. We use these results to provide some new bounds for inverses of positive triangular matrices with monotonic column entries. We also refine a result of Vecchio and Mallik. This new result is shown to be in a sense best possible under the given constraints.
Chapter 1: Introduction

Difference equations are most commonly used to describe the recursive progression of a process or numerical sequence. For our purposes (and in general) the expression of a difference equation will take the form

\[ x_n = f(x_{n-1}, x_{n-2}, ..., x_0), \quad n \geq 0 \]  

(1.1)

where \( f \) is a real-valued function. Expanding the recursive form defines a sequence on the positive integers.

The consideration of difference equations in terms of their resulting sequences on the positive integers allows for the idea of convolution, extended from the definition of convolution for smooth functions. In the study of differential equations [2] we find convolution defined via

\[ (f \ast g)(t) = f(t) \ast g(t) = \int_0^t f(\tau)g(t - \tau)d\tau = \int_0^t g(\upsilon)f(t - \upsilon)d\upsilon, \]  

(1.2)

where \( f \) and \( g \) are real-valued piecewise continuous functions and \( t \geq 0 \). This definition may be extended to the discrete case if we replace \( f \) and \( g \) with the sequences \( x = \{x_n\} \) and \( y = \{y_n\} \), \( n \) taking the place of the independent variable \( t \). We then set

\[ (x \ast y)_n = \{x_i\}_{i=0}^n \ast \{y_i\}_{i=0}^n = \sum_{j=0}^{n} x_{n-j}y_j = \sum_{j=0}^{n} x_n y_{n-j}, \quad n \geq 0. \]  

(1.3)
This transition from smooth to discrete finds a concrete example in discussions of the Volterra integrodifferential equation,

\[ x'(t) = Ax(t) + \int_0^t b(t-s)x(s)ds, \]  

(1.4)

and its associated discrete form, the Volterra difference equation,

\[ x_n = Ax_{n-1} + \sum_{l=0}^{n-1} b_{n-l}x_l. \]  

(1.5)

(See Elaydi [4].) The Volterra difference equation is an example of a difference equation of convolution type, where \( x_n \) is defined as the convolution of the sequence \( \{b_i\}_{i=0}^n \) with \( \{x_i\}_{i=0}^n \). This form is used in the research which motivates this thesis, and appears frequently in the literature. However, for ease of computation (and without loss of generality) we will principally use the comparable form

\[ x_n = \sum_{l=1}^{n} b_lx_{n-l}. \]  

(1.6)

When considering solutions to convolution-type difference equations, we seek to define properties of \( \{b_n\} \) that will lead to convergent, divergent or periodic behavior in \( \{x_n\} \). Elaydi [3] developed several conditions assuring asymptotic convergence to the zero solution of Equation (1.5) based on summation of the terms of \( \{b_n\} \). His paper makes particular conjectures on the sufficient and necessary nature of these conditions. In a recent paper of Elaydi and Vecchio [5] it was shown by counterexample that some of Elaydi’s earlier conditions are not necessary to guarantee convergence. This motivated our investigation of what stronger statements could be made about properties of \( \{b_n\} \) leading to convergence in \( \{x_n\} \). We present results from our initial
investigation in Chapter 2 of this thesis. We consider monotone \( \{b_n\} \), and apply results from renewal theory to prove conditions for the convergence of \( \{x_n\} \) irrespective of the sum of \( \{b_n\} \). We are also able to make explicit statements about the rate of convergence for many cases.

It is natural to extend these results to matrix analysis when one considers the general form of the inverse of a triangular \( n \times n \) matrix. For the matrix

\[
A_n = \begin{bmatrix}
    a_{1,1} \\
    a_{2,1} & a_{2,2} \\
    a_{3,1} & a_{3,2} & a_{3,3} \\
    \ddots & \ddots & \ddots \\
    a_{n,1} & \ldots & a_{n,n-1} & a_{n,n}
\end{bmatrix}, \tag{1.7}
\]

the entries of \( B_n = [b_{i,j}] = A_n^{-1} \) satisfy \( b_{j,j} = 1/a_{j,j} \) and

\[
b_{i,j} = \sum_{l=j}^{i-1} \frac{a_{i,l}}{a_{i,i}} b_{l,j} \tag{1.8}
\]

for \( 1 \leq j < i \leq n \). If we additionally define the sequence \( \{U_{i,j}\} \) via \( U_{i,j} = 0 \) if \( j > i \), \( U_{j,j} = 1/a_{j,j} \) and

\[
U_{i,j} = \frac{1}{a_{i,i}} - \sum_{l=j}^{i-1} \frac{a_{i,l}}{a_{i,i}} U_{l,j}, \tag{1.9}
\]

for \( i > j \), we have a sequence distinctly similar to that discussed in our examination of Volterra difference equations, and which proves susceptible to the same analysis. It is not difficult to show (see Lemma 2.2 in Chapter 3) that \( b_{i,j} = U_{i,j} - U_{i,j+1} \). We touch on this matrix analysis connection in the end of Chapter 2, and discuss it at length in the following chapters.
In Chapter 3 we further a result of Berenhaut, Morton and Fletcher [1] regarding triangular Toeplitz matrices. This enables us to determine a bound on $\|B_n\|_1 = \|A_n^{-1}\|_1$. For the specific case where $A$ has a constant diagonal, this result refines a previous theorem of Vecchio and Mallik [6]. In Chapter 4 we improve the theorem of Vecchio and Mallik by showing that certain summed terms in their upper bound on $\|B_n\|_1$ are unnecessary. Our bound may be considered in some sense optimal given the conditions on $A$. In fact, we show that if given

(i) $a_{i,j} \geq a > 0, \quad j = 1, \ldots, n, \quad i = j, \ldots, n,$

(ii) $a_{j,j} \geq a_{j+1,j} \geq \cdots \geq a_{n,j}, \quad j = 1, \ldots, n,$

and

$$\mathcal{A}_n(a) = \{ A = [a_{i,j}]_{n \times n} \mid A \text{ satisfies (1.7), (i) and (ii)} \} , \quad (1.10)$$

we have

$$\sup_{A \in \bigcup_{n \geq 1} \mathcal{A}_n(a)} \|A^{-1}\|_1 = 2/a. \quad (1.11)$$
Chapter 2: Equations of convolution type with monotone coefficients

CHAPTER 2

Equations of convolution type with monotone coefficients

Kenneth S. Berenhaut, Nathaniel G. Vish

The following paper is in revision for publication in *Journal of Difference Equations and Applications*. Stylistic variations are due to the requirements of the journal.
RESEARCH ARTICLE

Equations of convolution type with monotone coefficients

Kenneth S. Berenhaut\textsuperscript{a}, and Nathaniel G. Vish\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109.

(v3.4 released June 2008)

This paper studies convolution type linear difference equations with coefficients satisfying some monotonicity properties. Methods from renewal theory are employed to obtain easily verified conditions for asymptotic stability of the zero solution, in terms of the coefficient sequence. Explicit bounds and rates of convergence are also considered, and an application to norms of matrix inverses is included.

Keywords: difference equations; convolution; asymptotic stability; renewal sequences; exponential convergence; explicit bounds; matrix inequalities

AMS Subject Classification: 39A10; 39A11; 60K05; 15A09; 15A57

1. Introduction

In this paper, we consider solutions to recursive equations of the form

\[ x_n = \sum_{l=1}^{n} b_l x_{n-l}, \]

for \( n \geq 1 \), where \( x_0 \neq 0 \) is given and \( b_i \in \mathbb{R} \) for \( i \geq 1 \).

Systems of the form represented in (1) are known as hereditary in the sense that the present state can depend directly on all past states. If the sequence \( \{b_i\} \) forms a legitimate discrete probability distribution on \( \{1, 2, \ldots\} \), i.e. \( b_i \in [0,1] \) for \( i \geq 1 \) and \( \sum b_i = 1 \), then (1) is also known as a renewal equation and the resulting solution \( \{x_i\} \) as a renewal sequence (see for instance Feller [12]). Equations of the form in (1) also arise in matrix analysis (see Section 2, below) and in the theory of formal power series (see for instance [2] and the references therein).

As the discrete analogue of the Volterra integrodifferential equation

\[ x'(t) = Ax(t) + \int_{0}^{t} b(t-s)x(s)ds, \]

Equation (1) also appears in the literature in the form

\[ x_n = Ax_{n-1} + \sum_{l=0}^{n-1} b_{n-l} x_l, \]

*Corresponding author. Email: berenhks@wfu.edu
where $A \in \mathbb{R}$, (see [8–10] and the references therein). In [9] Elaydi provides the following two results regarding stability of solutions to (3).

**Theorem 1.1.** (Elaydi [9]). Suppose that the sequence $b_i$ does not change sign for $i \geq 1$, and

$$|A| + \sum_{i=1}^{\infty} b_i < 1. \quad (4)$$

Then the zero solution of (3) is asymptotically stable.

**Theorem 1.2.** (Elaydi [9]). Suppose that the sequence $\{b_i\}$ does not change sign for $i \geq 1$. Then the zero solution of (3) is not asymptotically stable if any one of the following conditions holds:

(i) $A + \sum_{i=1}^{\infty} b_i \geq 1$.

(ii) $A + \sum_{i=1}^{\infty} b_i \leq -1$ and $b_i > 0$ for some $i \geq 1$.

(iii) $A + \sum_{i=1}^{\infty} b_i < -1$ and $b_i < 0$ for some $i \geq 1$ and $\sum_{i=1}^{\infty} b_i$ is sufficiently small.

Theorems 1.1 and 1.2 are proved quite eloquently in [9] by employing Z-transforms and Rouche’s Theorem.

Unfortunately, results such as Theorems 1.1 and 1.2 which give criteria for stability or non-stability based on the sum of the coefficient sequence cannot be expected to fully capture the convergence behavior of solutions. Not only are there many instances of non-summable coefficient sequences which result in asymptotic stability (see Examples 1.11 and 4.3, below), but small shifts in value between coefficients can have a great effect on convergence properties as is seen by the following example.

**Example 1.3** Consider the sequence $\{b_i\}_{i=1}^{\infty} = \{0, -0.5, 0, -0.5, 0, 0, \ldots \}$. In this case, we have that the associated Z-transform has all zeroes inside the unit circle and the zero solution to (1) is asymptotically stable (see for instance [10]). Alternatively, if $\{b_i\}_{i=1}^{\infty} = \{-0.5, 0, -0.5, 0, 0, \ldots \}$, we have that for $x_0 = 1$,

$$x_k = C_1 \gamma_1^k + C_2 \gamma_2^k + \frac{(-1)^k}{2}, \quad (5)$$

for $k \geq 0$, where $|\gamma_1| = |\gamma_2| = 1/\sqrt{2}$ and $C_1, C_2$ are constants. Thus, here, $\{x_i\}$ is asymptotically two-periodic with limiting cycle $-0.5$ and $0.5$. Perhaps more importantly, in other cases, a small shift in coefficients (while maintaining constant sum) can have a dramatic effect on the rate of convergence to the zero solution when stability is maintained.

In [8] it was shown, by considering monotone and convex sequences $\{b_i\}_{i \geq 1}$, that the condition in (4) is, indeed, not necessary for asymptotic stability.

For other existing results on stability for solutions to (1) and (3) see for instance [1, 2, 5, 6, 8, 9, 12–14, 16] and the references therein.

**Theorem 1.4.** ([11, 12]) Suppose that $b_i \in [0, 1]$ for all $i$, $\sum_{i=1}^{\infty} b_i = 1$, $x_0 = 1$,
Equations of convolution type

\[ \mu = \sum_{i=1}^{\infty} ib_i \] and

\[ \gcd\{i \geq 1 : b_i > 0\} = 1, \] (6)

then

\[ \lim_{i \to \infty} x_i = x_\infty, \] (7)

where \( x_\infty = 1/\mu. \)

The next result regarding exponential convergence in (7), will be quite important in what follows.

**Theorem 1.5.** Suppose that \( b_i \in [0, 1] \) for all \( i, \sum_{i=1}^{\infty} b_i = 1, x_0 = 1 \) and \( b_1 \neq 1, \)
then

\[ |x_{i-1} - x_i| \leq r_i^{-i}, \] (8)

for \( i \geq 1, \) where

\[ r_i = \min_{1 \leq j \leq i: q_j > 0} \left\{ \frac{q_j-1}{q_j} \right\}, \] (9)

with \( q_0 = 1 \) and

\[ q_i = \sum_{j>i} b_j, \] (10)

for \( i \geq 1. \)

**Proof.** The proof follows the method employed in proving Theorem 2.1 in [5]. For some background on renewal theory see, for instance, [12] or [5].

First, note that some slight manipulation in (1) leads to the equation

\[ q_n = \sum_{k=1}^{n} (x_{k-1} - x_k)q_{n-k}, \] (11)

for \( n \geq 1, \) or equivalently

\[ g_n = q_n - \sum_{k=1}^{n-1} g_k q_{n-k}, \] (12)

where \( g_n = x_{n-1} - x_n \) for \( n \geq 1. \)

Now, suppose \( n \geq 1 \) is fixed. Since \( q_i \neq 0 \) and \( \{q_i\} \) is decreasing and positive, \( r_n \geq 1. \) If \( r_n = 1, \) then (8) follows directly from the fact that \( \{x_i\} \) is the sequence of renewal probabilities associated with the distribution \( \{b_i\} \) and hence takes on only values in \([0, 1]\). Thus, suppose \( r_n > 1, \) fix \( 1 < \rho < r_n \) and define \( \{f_i\} \) via

\[ f_i = q_{i-1}^\rho - q_i^\rho, \] (13)

for \( i \geq 1, \) where

\[ q_i^\rho = q_i \rho^i, \] (14)
for $0 \leq i \leq n$, and $q_i^* = 0$ for $i > n$. Note that $q_i^* = \sum_{j>i} f_j$ for $i \geq 1$. It is not difficult to show that by the definition of $r_n$, $\{q_i^*\}$ is a monotone null sequence and hence $\{f_i\}$ is a legitimate “lifetime” distribution on $\{1,2,\ldots\}$. Thus, in addition to (1) we also have that $\{x_i^*\}$ defined via $x_0^* = 1$ and

$$x_i^* = \sum_{j=1}^{i-1} f_j x_{i-j}^*$$  \hspace{1cm} (15)

is a renewal sequence. We thus have, via (12), that

$$g_i^* = q_i^* - \sum_{k=1}^{i-1} g_k^* q_{i-k}^*,$$  \hspace{1cm} (16)

for $n \geq 1$, where

$$g_i^* = x_{i-1}^* - x_i^*,$$  \hspace{1cm} (17)

for $i \geq 1$. Multiplying by $\rho^n$ in (12) and comparing with (16) gives, inductively, that $g_i^* = g_i \rho^i$ for $i \geq 1$. Since as a sequence of renewal probabilities $\{x_i^*\}$ satisfies $0 \leq x_i^* \leq 1$ for all $i \geq 0$, we have

$$|g_n| \leq \rho^{-n}.$$  \hspace{1cm} (18)

Letting $\rho$ tend to $r_n$ gives the inequality in (8). \hfill $\Box$

Note that the quantity $q_{i-1}/q_i$ is directly related to the hazard rates of the distribution $\{b_i\}$ (see for instance [5, 15]).

In [5], it was proven that in fact

$$|x_i - x_\infty| \leq r_{\min}^{-(i+1)},$$  \hspace{1cm} (19)

for $i \geq 0$, where

$$r_{\min} = \lim_{n \to \infty} r_n.$$  \hspace{1cm} (20)

By employing coupling-splitting arguments and minorization conditions, results similar to (19), with a slightly larger bound (see for instance Roberts and Polson [17] and Rosenthal [18]), have been obtained. For some additional results on exponential convergence see [1, 6, 13, 14, 16].

Here we will prove several results regarding stability of solutions to (1) under monotonicity constraints on the sequence $\{b_i\}$. Among our results are the following.

**Theorem 1.6.** Suppose that $\{b_i\}$ and $\{x_i\}$ satisfy (1) and $b_0 = -1$. If for some $n \geq 1$, $\{b_i\}_{0 \leq i \leq n}$ is a (not necessarily strictly) increasing sequence satisfying $b_i \in [-1,0]$, for $0 \leq i \leq n$, then

$$|x_n| \leq |x_0|r_{\min}^{-n},$$  \hspace{1cm} (21)
where

\[ r_n = \min_{1 \leq i \leq n : b_i < 0} \left\{ \frac{b_{i-1}}{b_i} \right\}. \quad (22) \]

**Theorem 1.7.** Suppose that \( \{b_i\} \) and \( \{x_i\} \) satisfy (1) and \( b_0 = -1 \). If \( \{b_i\} \) is a (not necessarily strictly) increasing sequence satisfying \( \lim_{i \to \infty} b_i = 0 \) and

\[ \gcd\{i \geq 1 : b_{i-1} < b_i\} = 1, \quad (23) \]

then \( \lim_{i \to \infty} x_i = 0 \).

Theorem 1.7 leads immediately to the following corollary.

**Corollary 1.8.** Suppose that \( \{b_i\} \) and \( \{x_i\} \) satisfy (1), \( b_0 = -1 \) and \( \lim_{i \to \infty} b_i = 0 \). The following hold.

(i) If \( \{b_i\} \) is a (not necessarily strictly) increasing sequence satisfying \( b_1 > -1 \), then \( \lim_{i \to \infty} x_i = 0 \).

(ii) If \( \{b_i\} \) is a strictly increasing sequence, then \( \lim_{i \to \infty} x_i = 0 \).

**Proof.** Note that if \( b_1 > -1 \) then \( b_0 = -1 < b_1 \) and the greatest common divisor in (23) is one. Hence (i) holds. To see (ii), note that in this case \( b_1 < b_2 \) and \( b_2 < b_3 \) and once again (23) holds. \( \square \)

**Remark 1.** With complete knowledge of the sequence \( \{b_i\} \) we can in principle compute the Z-transform or generating function associated with (1) and identify the largest geometric convergence rate possible for \( \{x_i\} \) (see for instance Heathcote [13] and Berenhaut and Lund [5, 6]). Firstly, root finding will not in general lead to an explicit bound as in (21). Second and perhaps more importantly, as mentioned in [9, 10], locating such zeroes can be very difficult in most problems.

We now present some simple, yet instructive examples.

**Example 1.9** Suppose that

\[ \{b_0, b_1, b_2, \ldots, \} = \{-1, -9/10, -7/10, -1/16, -1/32, -1/64, \ldots\}. \quad (24) \]

Applying Corollary 1.8 (i) or (ii), we have that \( \lim_{i \to \infty} x_i = 0 \).

Note that here \( b_0/b_1 = 10/9 \), \( b_1/b_2 = 9/7 \), \( b_2/b_3 = 56/5 \) and \( b_{i-1}/b_i = 2 \), for \( i \geq 4 \), and hence \( r_n = 10/9 \) for \( n \geq 1 \). Hence

\[ |x_n| \leq |x_0| \left( \frac{9}{10} \right)^n. \quad (25) \]

Computations suggest that the correct rate of convergence for \( \{x_i\} \) is in the neighborhood of \( 8/10 \). In fact the largest zero, \( z_0 \), of the Z-transform associated with (1) satisfies \( |z_0| \approx 0.7972233 \).

**Example 1.10** Suppose that

\[ \{b_0, b_1, b_2, \ldots, \} = \{-1, -9, -8, -7, 0, 0, 0, \ldots\}. \quad (26) \]

Applying Corollary 1.8 (i) or (ii), we have that \( \lim_{i \to \infty} x_i = 0 \).

Note that here \( b_0/b_1 = 10/9 \), \( b_1/b_2 = 9/8 \), and \( b_2/b_3 = 8/7 \), and hence as in
Example 1.9, $r_n = 10/9$ for $n \geq 1$. Thus, again

$$|x_n| \leq |x_0| \left(\frac{9}{10}\right)^n. \quad (27)$$

Computations here suggest that the correct rate of convergence for $\{x_i\}$ is larger than 0.88. In fact the largest zero, $z_0$, of the Z-transform associated with (1) satisfies $|z_0| \approx 0.8881555$.

**Example 1.11** Suppose that $\{b_i\}$ is the negative harmonic sequence defined via

$$\{b_0, b_1, b_2, \ldots, \} = \{-1, -1/2, -1/3, -1/4, -1/5, \ldots \}. \quad (28)$$

Applying Corollary 1.8 (i) or (ii), we have that $\lim_{i \to \infty} x_i = 0$.

Note that here $b_{i-1}/b_i = (i + 1)/i$, for $i \geq 1$, and hence $r_n = 1 + 1/n$ and

$$|x_n| \leq |x_0| \left(\frac{n}{n+1}\right)^n. \quad (29)$$

Unfortunately, the bound in (29) tends to $|x_0|/e > 0$ as $n$ tends to infinity, and hence is not useful in the limit.

We will also prove the following.

**Theorem 1.12.** Suppose that $\{b_i\}$ and $\{x_i\}$ satisfy (1). If the constant $\epsilon > 0$ is given, then the following hold.

(i) If $\{b_i\}$ is a (not necessarily strictly) increasing sequence which satisfies $b_i \in [-1, -\epsilon]$ for all $i \geq 1$, then $\lim_{i \to \infty} x_i = 0$.

(ii) If $\{b_i\}$ is a (not necessarily strictly) decreasing sequence which satisfies $b_i \in [-2 + \epsilon, -1]$ for all $i \geq 1$, then $\lim_{i \to \infty} x_i = 0$.

The remainder of the paper proceeds as follows. In Section 2, we give an application to matrix analysis, while Section 3 contains proofs of Theorems 1.6, 1.7 and 1.12. Some additional results are given in Section 4.

2. **An application to matrix analysis**

In this section we give an application to matrix analysis. In particular, consider $(n + 1) \times (n + 1)$ truncations of infinite lower triangular (real) Toeplitz matrices, i.e.

$$C_n = \begin{bmatrix} a_0 \\ a_1 & a_0 \\ a_2 & a_1 & a_0 \\ \vdots & \vdots & \vdots \\ a_n & \cdots & a_1 & a_0 \end{bmatrix}. \quad (30)$$

The following result was proven in [4].

**Theorem 2.1.** ([4]) Suppose that the sequence $\{a_i\}_{i \geq 0}$ satisfies

$$a_0 \geq a_1 \geq a_2 \geq \cdots a_n \geq a > 0, \quad (31)$$
for some constant $a$ and all $n$. Then

$$\|C_n^{-1}\|_1 \leq \frac{2}{a} \left( 1 - \xi(a, a_0)^{\frac{n}{2}} \right)$$

(32)

where $\xi$ is the inverse ratio defined via

$$\xi(x, y) = 1 - \frac{x}{y},$$

(33)

and, in particular,

$$\|C_n^{-1}\|_1 \leq \frac{2}{a},$$

(34)

independent of $a_0$ and $n$.

For related results see [3, 19, 20] and the references therein.

The proof of Theorem 2.1 in [4] involves employing the relationship

$$x_0 = \frac{1}{a_0}$$

and

$$x_n = \sum_{j=1}^{n} \left( \frac{a_j}{a_0} \right) x_{n-j},$$

(35)

for $n \geq 1$, where

$$C_n^{-1} = \begin{bmatrix}
x_0 & x_0 & x_0 \\
x_1 & x_1 & x_0 \\
x_2 & x_1 & x_0 \\
\vdots & \vdots & \vdots \\
x_n & \cdots & x_1 & x_0
\end{bmatrix}.$$  

(36)

The inequality in (34) implies Theorem 1.12 (i) directly, although we will provide an alternate proof, below. More importantly, employing Corollary 1.8 (ii), we have the following.

**Theorem 2.2.** Suppose that the sequence $\{a_i\}_{i \geq 0}$ satisfies

$$a_0 \geq a_1 > a_2 > \cdots a_n \geq 0,$$

(37)

then the entries in the inverse matrix $C_n^{-1}$ as in (36) satisfy

$$|x_n| \leq \frac{1}{a_0} r_n^{-n},$$

(38)

where

$$r_n = \min_{1 \leq i \leq n; a_i > 0} \left\{ \frac{a_i-1}{a_i} \right\}.$$  

(39)

In particular, if $1 < r = \lim_{n \to \infty} r_n$, then

$$\|C_n^{-1}\|_1 \leq \frac{1}{a_0} \frac{r}{r - 1},$$

(40)
for \( n \geq 1 \).

We now turn to proofs of Theorems 1.6, 1.7 and 1.12.

### 3. Main Results

For given sequences \( \{b_i^*\}_{i \geq 1} \) and \( \{x_i^*\}_{i \geq 0} \) satisfying \( x_0^* = 1 \) and

\[
x_i^* = \sum_{j=1}^{i-1} b_j^* x_{i-j}^*,
\]

(41)

set \( q_0^* = 1 \) and

\[
q_i^* = 1 - \sum_{j=1}^{i} b_j^*,
\]

(42)

for \( i \geq 1 \).

It is quite easy to show, as in the proof of Theorem 1.5, that for \( m \geq 1 \),

\[
g_m^* = q_m^* - \sum_{j=1}^{m-1} q_j^* g_{m-j}^*,
\]

(43)

where \( g_0^* = -1 \) and

\[
g_i^* = x_{i-1}^* - x_i^*,
\]

(44)

for \( 1 \leq i \leq m \).

We are now in a position to prove Theorems 1.6, 1.7 and 1.12.

**Proof of Theorem 1.6.** Set \( b_{n+1}^* = -b_n \), \( b_i^* = 0 \) for \( i > n + 1 \), and

\[
b_i^* = b_i - b_{i-1},
\]

(45)

for \( 1 \leq i \leq n \). We have that \( \{b_i^*\} \) satisfies the assumptions of Theorem 1.5, and hence employing (44),

\[
|g_n^*| \leq (r_n^*)^{-n},
\]

(46)

where

\[
r_n^* = \min_{1 \leq i \leq n; q_i^* > 0} \left\{ \frac{q_{i-1}^*}{q_i^*} \right\}.
\]

(47)

Since \( b_0 = -1 \), (45) and (42) give, inductively, that \( b_i = -q_i^* \) for \( 0 \leq i \leq n \). Comparing (43) with (1) then gives

\[
|x_i| = |g_i^*||x_0|
\]

(48)
for $0 \leq i \leq n$. The inequality in (21) then follows from (46).

Proof of Theorem 1.7. Here, set

$$b_i^* = b_i - b_{i-1},$$

for $1 \geq 1$. We have that $\{b_i^*\}$ satisfies the hypotheses of Theorem 1.4 and hence by (41), $\{x_i^*\}$ is a convergent sequence. By (44) $\{q_i^*\}$ is then a null sequence. As in the proof of Theorem 1.6, $b_i = -q_i^*$ for $i \geq 0$. Hence, (48) holds for $i \geq 0$, and the theorem is proven. □

Proof of Theorem 1.12. Defining $\{b_i^*\}$ as in (49), under the assumptions in (i) and (ii), we have that $b_i^*$ does not change sign for $i \geq 1$ and $|\sum b_i^*| < 1$. By Theorem 1.1, $\{x_i^*\}$ converges, and hence, employing (44), $\{g_i^*\}$ is a null sequence. The results in (i) and (ii) follow upon noting, as in the proof of Theorem 1.7, that $b_i = -q_i^*$ for $i \geq 0$, and employing (48). □

4. Concluding remarks

In this note, we have given some results, both qualitative and quantitative, regarding convergence of solutions to linear difference equations of convolution type, when the coefficient sequence satisfies some monotonicity properties. It is of course possible to continue along the path of employing a theorem regarding convergence to obtain another on convergence when the original sequence $\{b_i\}$ is replaced with $\{-q_i\}$ where $q_i$ is defined as in (10). The result can be formulated as follows.

**Theorem 4.1.** Suppose that $\{b_i^*\}$ and $\{x_i^*\}$ are related by (41) and define $\{q_i^*\}$ via $q_0^* = 1$ and $q_i^* = 1 - \sum_{1 \leq j \leq i} b_j^*$, for $i \geq 1$. If $\{x_i^*\}$ is convergent then the resulting sequence $\{x_i\}$ defined via

$$x_n = \sum_{j=1}^{n} (-q_j^*)x_{n-j},$$

for $n \geq 1$, with given $x_0 \in \mathbb{R}$, converges to zero. Moreover if

$$|x_i^*| \leq C s^i,$$

for some $C, s > 0$ and all $0 \leq i \leq n$, then

$$|x_i| \leq C|x_0| \left(1 + \frac{1}{s}\right)s^{i-1},$$

for $1 \leq i \leq n$.

Proof. The result follows as in the proofs of Theorems 1.6, 1.7 and 1.12. Note that by (43) and (50),

$$|x_i| = |g_i^*|x_0| = |x_i^* - x_{i-1}^*| |x_0|$$

and hence if $\{x_i^*\}$ is convergent then $\{x_i\}$ is a null sequence. In addition, if (51) holds, then (53) implies (52) and the result is proven. □

As an example, employing Corollary 1.8 (i) and Theorem 4.1, we have the following theorem.
Theorem 4.2. Suppose that \( \{b_i\}_{i \geq -1} \) and \( \{x_i\}_{i \geq 0} \) satisfy (1) with 

1. \( b_{-1} = 0, b_0 = -1 \) and \(-2 < b_1 < -1\),
2. \( \Delta b_i = b_{i+1} - b_i \leq 0 \), for \(-1 \leq i \leq n - 1\),
3. \( \Delta^2 b_i = (b_{i+2} - b_{i+1}) - (b_{i+1} - b_i) \geq 0 \), for \(-1 \leq i \leq n - 2\).

Then,

\[
|x_n| \leq |x_0| \left( 1 + \frac{1}{r_n} \right) r_n^{-(n-1)},
\]

for \( n \geq 1 \), where with \( b_{-1} = 0 \)

\[
r_n = \min_{0 \leq i \leq n-1} \left\{ \min_{b_{i+1} < b_i} \left\{ \frac{b_i - b_{i-1}}{b_{i+1} - b_i} \right\} \right\}.
\]

In addition, if \( \lim_{i \to \infty} b_{i+1} - b_i = 0 \) then \( \lim_{i \to \infty} x_i = 0 \).

Proof. Here we employ Theorem 4.1 for the set of coefficients \( \{b_i^*\}_{i=1}^\infty = \{b_1 - b_0, b_2 - b_1, \ldots\} \). By the given assumptions this sequence satisfies the hypotheses of Corollary 1.8, and hence the result follows. \( \square \)

Example 4.3 Suppose that

\[
\{b_{-1}, b_0, b_1, b_2, \ldots\} = \{0, -1, -1.9, -2.7, -3.3, -3.3, -3.3, \ldots\}. \tag{56}
\]

Here \( b_0 - b_{-1} = -1, b_1 - b_0 = -0.9, b_2 - b_1 = -0.8, b_3 - b_2 = -0.6 \) and \( b_{i+1} - b_i = 0 \) for \( i \geq 3 \). We have that the assumptions of Theorem 4.2 are satisfied and the zero solution is asymptotically stable. In addition \( r_n = 10/9 \) for all \( n \geq 2 \) and hence, 

\[
|x_n^*| \leq (9/10)^i \text{ for } i \geq 0.
\]

Thus,

\[
|x_n| \leq |x_0| \frac{19}{10} \left( \frac{9}{10} \right)^{n-1}
\]

for \( n \geq 0 \).

Acknowledgements

We are very thankful to referees for comments and insights that substantially improved this manuscript.

References

REFERENCES


Chapter 3: A 1-norm Bound for Inverses of Triangular Matrices with Monotone Entries

CHAPTER 3

A 1-norm Bound for Inverses of Triangular Matrices with Monotone Entries

Kenneth S. Berenhaut, Richard T. Guy, Nathaniel G. Vish

The following paper has appeared in the *Banach Journal of Mathematical Analysis*. Stylistic variations are due to the requirements of the journal.
A 1-NORM BOUND FOR INVERSES OF TRIANGULAR MATRICES WITH MONOTONE ENTRIES

KENNETH S. BERENHAUT\textsuperscript{1,*}, RICHARD T. GUY\textsuperscript{1} AND NATHANIAL G. VISH\textsuperscript{1}

Abstract. This paper provides some new bounds for 1–norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality of Vecchio and Mallik in the case of constant diagonal. The results are shown to be in a sense best possible under the given constraints. En route some partial order inequalities are obtained.

1. Introduction

This paper provides some new bounds for 1–norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality of Vecchio and Mallik [11] in the case of constant diagonal. We refer the reader to Vecchio [10] and Vecchio and Mallik [11] (and the reference therein) for discussion of applications particularly those to stability analysis of linear methods for solving Volterra integral equations. Other references on the topic include [3]–[7] and [9].

The matrices of interest here are \( n \times n \) truncations of infinite lower triangular (real) matrices, i.e.
The following result was proven in [11].

**Theorem 1.1.** Assume that
(i) $a_{i,j} \geq a > 0$, $j = 1, \ldots, i$, $i = 1, \ldots, n$,
(ii) $a_{i,i} \geq a_{i+1,i} \geq \cdots \geq a_{n,i}$, $i = 1, \ldots, n$,
and let
\begin{equation}
    a_{\min} = \min_{i=1,\ldots,n} \{a_{i,i}\},
\end{equation}
and $B_n = [b_{i,j}]$ be the inverse of the lower triangular matrix $A_n$. Then
\begin{equation}
    \|B_n\|_1 \leq \frac{1}{a_{\min}} + \frac{2}{a}.
\end{equation}

The result in (3) was first proven in the case of triangular Toeplitz matrices in [10] and improved to the following in [2].

**Theorem 1.2.** Suppose that the sequence $\{a_i\}_{i \geq 0}$ satisfies
\begin{equation}
    a_0 \geq a_1 \geq a_2 \geq \cdots a_n \geq a > 0,
\end{equation}
for some constant $a$ and all $n$ and
\begin{equation}
    C_n = \begin{bmatrix}
    a_0 & & \\
    a_1 & a_0 & \\
    & a_2 & a_1 & a_0 \\
    & & \ddots & \ddots & \ddots \\
    & & & a_n & \cdots & a_1 & a_0
    \end{bmatrix}.
\end{equation}

Then
\begin{equation}
    \|C^{-1}_n\|_1 \leq \frac{2}{a} \left(1 - \rho(a, a_0) \left\lceil \frac{a}{a_0} \right\rceil \right),
\end{equation}
where $\rho$ is the inverse ratio defined via
\begin{equation}
    \rho(x, y) = 1 - x/y,
\end{equation}
and, in particular
\begin{equation}
    \|C^{-1}_n\|_1 \leq \frac{2}{a},
\end{equation}
independent of $a_0$ and $n$. 

\begin{equation}
    \begin{pmatrix}
    a_{1,1} \\
    a_{2,1} & a_{2,2} \\
    a_{3,1} & a_{3,2} & a_{3,3} \\
    & \ddots & \ddots & \ddots \\
    a_{n,1} & \cdots & a_{n,n-1} & a_{n,n}
    \end{pmatrix}
\end{equation}

\begin{equation}
    \begin{array}{c}
    a_{1,1} \\
    a_{2,1} & a_{2,2} \\
    a_{3,1} & a_{3,2} & a_{3,3} \\
    & \ddots & \ddots & \ddots \\
    a_{n,1} & \cdots & a_{n,n-1} & a_{n,n}
    \end{array}
\end{equation}
Here, we extend Theorem 1.2 (to non-Toeplitz matrices) and refine Theorem 1.1 in the case of constant diagonal. In particular we will prove the following.

**Theorem 1.3.** Assume that the hypotheses of Theorem 1.1 are satisfied and in addition that

\[ a_{1,1} \leq a_{2,2} \leq \cdots \leq a_{n,n}. \]  

(9)

Then

\[ \|B_n\|_1 \leq \frac{2}{a} \left( \frac{a_{n,n}}{a_{1,1}} \right) \left( 1 - \frac{\rho(a, a_{n,n}) \left\lceil \frac{n}{2} \right\rceil + \rho(a, a_{n,n}) \left\lfloor \frac{n}{2} \right\rfloor}{2} \right). \]  

(10)

In particular, if

\[ a_{1,1} = a_{2,2} = \cdots = a_{n,n} = a^*, \]  

(11)

then

\[ \|B_n\|_1 \leq \frac{2}{a} \left( 1 - \frac{\rho(a, a^*) \left\lceil \frac{n}{2} \right\rceil + \rho(a, a^*) \left\lfloor \frac{n}{2} \right\rfloor}{2} \right) \]  

(12)

and hence

\[ \|B_n\|_1 < \frac{2}{a}, \]  

(13)

independent of \( a^* \).

Note that triangular matrices satisfying (11) arise in the study of linear groups (see for instance [8]) and are particularly important in the theory of matrix decompositions.

The inequality in (12) is in a sense best possible. In particular, for \( 0 < a < a^* \), set

\[ \mathcal{A}_n(a, a^*) = \{ A = [a_{i,j}]_{n \times n} \mid A \text{ satisfies (1), (i), (ii) and (11)} \}. \]  

(14)

We have the following theorem regarding optimality.

**Theorem 1.4.** For \( 0 < a < a^* \),

\[ \min_{A \in \mathcal{A}_n(a, a^*)} \|A^{-1}\|_1 = \frac{2}{a} \left( 1 - \frac{\rho(a, a^*) \left\lceil \frac{n}{2} \right\rceil + \rho(a, a^*) \left\lfloor \frac{n}{2} \right\rfloor}{2} \right). \]  

(15)

**Proof.** We need to show that the bound in (12) is attained. To that end, suppose \( a_{i,j} = a^* > 0 \) for \( i - j \in \{0, 1\} \) and \( a_{i,j} \equiv a \) otherwise. It is easy to verify in this case, that for \( 1 \leq j \leq i \leq n \),

\[ b_{i,j} = (-1)^{i-j} \frac{1}{a^*} \left( 1 - \frac{a}{a^*} \right)^{\left\lfloor \frac{i-j}{2} \right\rfloor}, \]  

(16)
and hence,

\[ \| A_n^{-1} \|_1 = \sum_{i=1}^{n} |b_{i,1}| \]

\[ = \sum_{i=1}^{n} \frac{1}{a^*} \left( 1 - \frac{a}{a^*} \right)^{\left\lfloor \frac{i-1}{2} \right\rfloor} = \sum_{i=0}^{n-1} \frac{1}{a^*} \left( 1 - \frac{a}{a^*} \right)^{\left\lfloor \frac{i}{2} \right\rfloor} \]

\[ = \frac{1}{a^*} \left( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor-1} \left( 1 - \frac{a}{a^*} \right)^i + \sum_{i=0}^{\left\lceil \frac{n}{2} \right\rceil-1} \left( 1 - \frac{a}{a^*} \right)^i \right) \]

\[ = \frac{2}{a} \left( 1 - \frac{\rho(a, a^*) \left\lceil \frac{n}{2} \right\rceil}{2} + \frac{\rho(a, a^*) \left\lfloor \frac{n}{2} \right\rfloor}{2} \right). \tag{17} \]

\[ \square \]

Note also that

\[ \inf_{A \in \bigcup_{n \geq 1} A_n(a, a^*)} \| A^{-1} \|_1 = 2/a. \tag{18} \]

The reader is referred to [1] for some discussion of bounds for inverses of matrices of the form in (1) when the condition of monotonicity within columns is replaced with that within rows.

2. Preliminaries and notation

In order to prove Theorem 1.3, consider the partial order on the set \( V_{b,a} \) of (arbitrary length) tuples \((a_1, a_2, \ldots, a_k)\) with

\[ b \geq a_1 > a_2 > \cdots > a_k = a \tag{19} \]

defined via

\[ \mathbf{v} \prec \mathbf{z} \text{ if } \mathbf{z} \text{ is a suffix of } \mathbf{v} \tag{20} \]

where \( \mathbf{z} = (z_1, z_2, \ldots, z_k) \) is a suffix of \( \mathbf{v} = (v_1, v_2, \ldots, v_m) \) if \( m > k \) and \( \mathbf{v} = (v_1, \ldots, v_{m-k}, z_1, z_2, \ldots, z_k) \). For convenience, if \( \mathbf{w} = (w_1, w_2, \ldots, w_r) \) we will write the \( r + k \)-tuple \((w_1, w_2, \ldots, w_r, z_1, z_2, \ldots, z_k)\) as \((\mathbf{w}; \mathbf{z})\). In addition, denote the length of \( \mathbf{v} \) by \( l(\mathbf{v}) = k \). The value \( v_1 \) will be referred to as the initial value of \( \mathbf{v} \).

For a triangular double sequence \( \{d_{i,j}\}_{j<i<n} \) satisfying \( 0 \leq d_{i,j} < 1 \) for \( j < i < n \) and

\[ \sum_{i=j+1}^{n} d_{i,j} \leq x < 1, \quad j = 1, 2, \ldots, n - 1 \tag{21} \]
define the function $D$ via
\[ D(v) = d_{v_1,v_{k-1}} \cdot d_{v_{k-1},v_{k-2}} \cdots d_{v_3,v_2} \cdot d_{v_2,v_1} \tag{22} \]
for $v = (v_1, v_2, \ldots, v_k)$.

Note that it follows directly from the definition of $D$, the inequality in (21) and the non-negativity of \{d_{i,j}\} that $D(v) < D(z)$ for $v < z$.

**Lemma 2.1.** Consider a set of tuples \{v_1, v_2, \ldots, v_k\}. If $v_i < z$ for $1 \leq i \leq k$ and $v_i \not< v_j$ for $i \neq j$ (i.e. \{v_1, v_2, \ldots, v_k\} forms an antichain that is bounded above by $z$) then
\[ D(z) \geq D(v_1) + D(v_2) + \cdots + D(v_k). \tag{23} \]

**Proof.** Let $z_2$ be the least upper bound for \{v_1, v_2, \ldots, v_k\}, i.e. $z_2 = \min\{w \preceq z : v_i \preceq w, 1 < i < k\}$. Clearly, $z_2 \preceq z$. We will show that
\[ \sum_{i=1}^{k} D(v_i) \leq D(z_2). \tag{24} \]

The result is immediate for $k = 1$. Hence suppose (24) holds for $1 \leq k < K$.

Now, suppose that there exists a $z_3 < z_2$ and a set $S \subset \{1, 2, \ldots, K\}$ such that $2 \leq ||S|| \leq K - 1$, $v_i < z_3$ if $i \in S$, and $v_i \not< z_3$, if $i \in S^c$.

then by induction, we have
\[ \sum_{i=1}^{K} D(v_i) \leq D(z_3) + \sum_{i \in S^c} D(v_i). \tag{25} \]

Considering the set \{z_3\} $\cup \{v_i : i \in S^c\}$ and applying induction again we have the inequality in (24).

Otherwise $v_i$ is of the form $v_i = (w_i; (t_i); z_3)$, $i = 1, 2, \ldots, k$, where $t_i \neq t_j$ for $l \neq j$ and $z_3 = (z_3,1, \ldots, z_3,k) \preceq z_2$. In this case, by (21),
\[ \sum_{i=1}^{K} D(v_i) \leq \sum_{i=1}^{K} D((t_i); z_3) = \sum_{i=1}^{K} d_{t_i,z_3,1} D(z_3) = D(z_3) \sum_{i=1}^{K} d_{t_i,z_3,1} \leq D(z_3) \tag{26} \]

and the proof is complete.

The following lemma will be crucial.

**Lemma 2.2.** For fixed $s \geq 1$, set $S_{i,s} = 0$ for $i < s$, $S_{s,s} = 1$ and for $s + 1 \leq m \leq n$, inductively,
\[ S_{m,s} = \sum_{i=s}^{m-1} d_{m,i} S_{i,s}. \tag{27} \]

Then, for $Q \subseteq \{s + 1, \ldots, n\}$, we have
\[ \sum_{i \in Q} S_{i,s} \leq x_s + x_s^2 + \cdots + x_s\|Q\|, \]  
where \( x_s = \max_{t=s}^{n-1} \sum_{i=t+1}^n d_{i,t} \).

**Proof.** Note that it follows from straightforward induction that for \( m > s \),
\[ S_{m,s} = \sum_{v=(m,\ldots,s) \in V_{m,s}} D(v). \]  
(29)

Note that in (29), the summation is over all tuples \( v = (v_1, v_2, \ldots, v_l(v)) \) with 
\[ m = v_1 > v_2 > \cdots > v_l(v) = s. \]  
(30)

Now, define
\[ L_{m,s}^k = \sum_{v \in V_{m,s} \atop l(v) = k+1} D(v) \]  
(31)

We will show inductively that
\[ L_{m,s}^k \leq x_s^k. \]  
(32)

First note that by (21) and the definition of \( x_s \),
\[ L_{m,s}^1 = d_{s+1,s} + d_{s+2,s} + d_{s+3,s} + \cdots + d_{m,s} \leq x_s. \]  
(33)

Thus assume that (32) is true for \( k < K \). Then, since \( x_1 \geq x_2 \geq \cdots \geq x_n \),
\[ L_{m,s}^K = \sum_{i=s+1}^m d_{i,s} L_{m,i}^{K-1} \leq \sum_{i=s+1}^m d_{i,s} x_i^{K-1} \leq x_s^{K-1} \sum_{i=s+1}^m d_{i,s} \leq x_s^K. \]  
(34)

Now, define the sets
\[ R_1 = \{ v \in V_{m,s} \mid 2 \leq l(v) \leq \|Q\| + 1 \} \]  
(35)
and
\[ R_2 = \{ v \in V_{m,s} \mid v = (i,\ldots,s), \ i \in Q \} \]  
(36)
and consider the quantity
\[ H_Q = \sum_{k=1}^{\|Q\|} L_{m,s}^k - \sum_{i \in Q} S_{i,s} \]  
(37)
\[ = \sum_{v \in R_1} D(v) - \sum_{v \in R_2} D(v). \]  
(38)

We will prove that for all sets \( Q \), \( H_Q \geq 0 \). The result will then follow from (38) and the inequality in (32).

We define the following scheme for matching elements \( z \) in \( R_1 \) with (possibly empty) subsets \( S(z) \) of \( R_2 \) such that \( D(z) > \sum_{v \in S(z)} D(v) \) and \( \{S(z) \mid z \in R_1 \} \) is a partition of \( R_2 \). In particular for \( 2 \leq t \leq n \), set
\[ J_t = \{ v \in R_1 : l(v) = t \}, \]  
(39)
and recursively in $t \geq 2$, for $z \in J_t$ let
\[ S(z) = \{ v \in R_2 | v \text{ is a maximal element in the set } W(z) \}. \] (40)

where
\[ W(z) = \{ w \in R_2 | w \preceq z \text{ and } w \notin \bigcup_{v \in z} S(v) \}. \] (41)

Here, again, the maximality in (40) is with respect to the given partial order on $V_{m,s}$.

Now, fix $z \in J_t$ for some $2 \leq t \leq n$ and suppose $\{ v_1, v_2 \} \subset S(z)$ with $v_1 \not= v_2$. The fact that $v_1 \not\succ v_2$ and $v_2 \not\succ v_1$ follows from the maximality in (40). We then have that Lemma 2.1 is applicable and
\[ D(z) \geq \sum_{v \in S(z)} D(v), \] (42)
as required. In addition, by the definition of $W$ we have that the sets $S(z), z \in R_1$ are pairwise disjoint. To see that $R_2 \subset \bigcup_{z \in R_1} S(z)$, first suppose $v \in R_2$.

Let $K_v$ be a maximal chain in $V_{m,s}$ such that $v \in K_v$ and set $T_1 = K_v \cap R_1 = \{ z_1, z_2, \ldots, z_r \}$ and $T_2 = K_v \cap R_2 = \{ v_1, v_2, \ldots, v_q \}$, where $v_1 \succ v_2 \succ \cdots \succ v_q$ and $z_1 \succ z_2 \succ \cdots \succ z_r$. Note that $\| T_1 \| = \| Q \|$ and $\| T_2 \| \leq \| Q \|$ (since the only possible initial values for tuples are those in $Q$) and by (40), $v_i \in S(z_i)$ for $1 \leq i \leq r$ and in particular $v \in \bigcup_{z \in R_1} S(z)$. Since $\bigcup_{z \in R_1} S(z) \subset R_2$ by (40), the result is proven. \hfill \Box

3. Proof of the main theorem

In this section we prove Theorem 1.3.

First note that the lower triangular matrix $B_n = [b_{i,j}] = A_n^{-1}$ satisfies $b_{s,s} = 1/a_{s,s}$ and
\[ b_{m,s} = \sum_{j=s}^{m-1} \alpha_{m,j} b_{j,s}, \] (43)
for $1 \leq s < m \leq n$, where $\alpha_{m,j} = (a_{m,j}/a_{m,m})$ for $1 \leq j \leq m \leq n$ (see for instance [1]).

Define $h_{i,j} = a_{j,i} b_{i,j}$ for $1 \leq j \leq i \leq n$, so that $h_{s,s} = 1$ and for $1 \leq s < m \leq n$,
\[ h_{m,s} = \sum_{j=s}^{m-1} \alpha_{m,j} h_{j,s}. \] (44)

We have the following lemma (contrast with Equation (2.3) in [11]).

Lemma 3.1. Suppose that $[a_{i,j}]$ satisfies the hypotheses of Theorem 1.3. Then
\[ h_{i,j} = S_{i,j} - S_{i,j+1}, \] (45)
for $1 \leq j \leq i \leq n$, where $\{S_{i,j}\}$ is as in (27) for the nonnegative double sequence $\{d_{i,j}\}$ defined via
\[
d_{m,j} = \alpha_{m-1,j} - \alpha_{m,j},
\]
for $1 \leq j < m \leq n$. In addition, (21) is satisfied with
\[
x_s = \max_{t=s,\ldots,n-1} \sum_{i=t+1}^{n} d_{i,t} \leq 1 - \frac{a}{a_n} = x.
\]

**Proof.** First, note that by (46), (ii) and (9)
\[
d_{m,j} = \alpha_{m-1,j} - \alpha_{m,j} = \frac{a_{m-1,j}}{a_{m,m-1}} - \frac{a_{m,j}}{a_{m,m}} \geq 0,
\]
and
\[
\sum_{m=j+1}^{n} d_{m,j} = \alpha_{j,j} - \alpha_{n,j} = 1 - \frac{a_{n,j}}{a_{n,n}} \leq 1 - \frac{a}{a_n} < 1.
\]
In addition, for $s + 2 \leq m \leq n$, $s = 1, 2, \ldots, n$,
\[
h_{m,s} - h_{m-1,s} = \sum_{j=s}^{m-1} -\alpha_{m,j}h_{j,s} + \sum_{j=s}^{m-2} \alpha_{m-1,j}h_{j,s}
\]
\[
= \sum_{j=s}^{m-2} (\alpha_{m-1,j} - \alpha_{m,j})h_{j,s} - \alpha_{m,m-1}h_{m,m-1},
\]
and hence since $d_{m,m-1} = \alpha_{m-1,m-1} - \alpha_{m,m-1} = 1 - \alpha_{m,m-1}$,
\[
h_{m,s} = \sum_{j=s}^{m-2} d_{m,j}h_{j,s} + (1 - \alpha_{m,m-1})h_{m,m-1} = \sum_{j=s}^{m-1} d_{m,j}h_{j,s}.
\]
In addition,
\[
S_{m,s} - S_{m,s+1} = \sum_{i=s}^{m-1} d_{m,i}S_{i,s} - \sum_{i=s+1}^{m-1} d_{m,i}S_{i,s+1}
\]
\[
= d_{m,s}S_{s,s} + \sum_{i=s+1}^{m-1} d_{m,i}(S_{i,s} - S_{i,s+1})
\]
\[
= d_{m,s}(S_{s,s} - S_{s,s+1}) + \sum_{i=s+1}^{m-1} d_{m,i}(S_{i,s} - S_{i,s+1})
\]
\[
= \sum_{i=s}^{m-1} d_{m,i}(S_{i,s} - S_{i,s+1}),
\]
since $S_{s,s+1} = 0$. 

Comparing (51) and (52) and noting that \( h_{s,s} = 1 = S_{s,s} - S_{s,s+1} \) and \( h_{s+1,s} = -\alpha_{s+1,s} = (1 - \alpha_{s+1,s}) - 1 = d_{s+1,s} S_{s,s} - 1 = S_{s+1,s} - S_{s+1,s+1} \), the result follows. \( \square \)

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3.** Employing Lemma 3.1 and the definition of \( \{h_{i,j}\} \), we have

\[
\|A_n^{-1}\|_1 = \|B_n\|_1 = \max_{1 \leq j \leq n} \sum_{i=j}^n |b_{i,j}| = \max_{1 \leq j \leq n} \sum_{i=j}^n \left| \frac{1}{a_{i,j}} (S_{i,j} - S_{i,j+1}) \right|. \tag{53}
\]

Now, fix \( 1 \leq j \leq n \). We have, by the nonnegativity of \( \{S_{i,j}\} \), that

\[
\sum_{i=j}^n |S_{i,j} - S_{i,j+1}| = |S_{j,j} - S_{j,j+1}| + |S_{j+1,j} - S_{j+1,j+1}| + \sum_{i \in Q_1} (S_{i,j} - S_{i,j+1}) + \sum_{i \in Q_1^c} (S_{i,j+1} - S_{i,j}) \leq |S_{j,j} - S_{j,j+1}| + |S_{j+1,j} - S_{j+1,j+1}| + \sum_{i \in Q_1} S_{i,j} + \sum_{i \in Q_1^c} S_{i,j+1}, \tag{54}
\]

where \( Q_1 = \{j + 2 \leq i \leq n | S_{i,j} > S_{i,j+1}\} \).

Noting that \( S_{j,j} = 1, S_{j,j+1} = 0, S_{j+1,j} = d_{j+1,j} < 1 \) and \( S_{j+1,j+1} = 1 \), we have from (54) that

\[
\sum_{i=j}^n |S_{i,j} - S_{i,j+1}| \leq 2 + \sum_{i \in Q_1} S_{i,j} + \sum_{i \in Q_1^c} S_{i,j+1}. \tag{55}
\]

Letting \( y = \|Q_1\| \leq n - j - 1 \), recalling \( x_j \leq 1 - a/a_{n,n} = x < 1 \) and employing Lemma 2.2 gives

\[
\sum_{i=j}^n |S_{i,j} - S_{i,j+1}| \leq (1 + x + \cdots + x^y) + (1 + x + \cdots + x^{n-j-1-y}) \leq \frac{1 - x^{y+1}}{1 - x} + \frac{1 - x^{n-(y+1)}}{1 - x} \leq \frac{2 - (x^{y+1} + x^{n-(y+1)})}{1 - x}. \tag{56}
\]
By the convexity of the function $f$ defined via $f(t) = x^t$, we have that $x^{y+1} + x^{n-(y+1)} \geq x^{\lceil n/2 \rceil} + x^{\lfloor n/2 \rfloor}$. Thus, returning to (53), we obtain

$$\|A_n^{-1}\|_1 \leq \frac{2}{\min_i \{a_{i,i}\}} \frac{1 - x^{\lceil n/2 \rceil} + x^{\lfloor n/2 \rfloor}}{1 - x}$$

$$= \frac{2}{a_{1,1}} \left( 1 - \frac{(1-a/a_{n,n})\lceil \frac{n}{2} \rceil + (1-a/a_{n,n})\lfloor \frac{n}{2} \rfloor}{a/a_{n,n}} \right).$$

(57)

In the case when $a_{i,i} = a^*$ for all $i$, (57) gives

$$\|A_n^{-1}\|_1 \leq \frac{2}{a} \left( 1 - \frac{(1-a/a^*)\lceil \frac{n}{2} \rceil + (1-a/a^*)\lfloor \frac{n}{2} \rfloor}{2} \right),$$

(58)

as required. □

References


1 Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109.
E-mail address: benhec@wfu.edu, guyrt@wfu.edu and vishng@wfu.edu
Chapter 4: An Optimal Bound for Inverses of Triangular Matrices with Monotone Entries

Kenneth S. Berenhaut, Richard T. Guy, Nathaniel G. Vish

The following paper has appeared in the *Journal of Linear and Multilinear Algebra*.

Stylistic variations are due to the requirements of the journal.
RESEARCH ARTICLE

An Optimal Bound for Inverses of Triangular Matrices with Monotone Entries

Kenneth S. Berenhaut\textsuperscript{a}\ast, Richard T. Guy\textsuperscript{a} and Nathaniel G. Vish\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109.

\(v3.2\) released May 2008

This paper provides a new bound for 1–norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality established in A. Vecchio and R. K. Mallik, Bounds on the inverses of nonnegative lower triangular Toeplitz matrices with monotonicity properties, \textit{Linear and Multilinear Algebra}, \textbf{55} (2007), no. 4, pp. 365–379. The results are shown to be in a sense best possible under the given constraints.

\textbf{Keywords:} inverse matrix, monotone entries, triangular matrix, recurrence relations, optimal bound

\textbf{AMS Subject Classification:} 15A09; 39A10; 15A57; 15A60

1. Introduction

This note provides a new bound for 1–norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality of Vecchio and Mallik [11]. We refer the reader to Vecchio [10] and Vecchio and Mallik [11] (and the references therein) for discussion of applications particularly those to stability analysis of linear methods for solving Volterra integral equations. Other references on the topic include [4]–[7] and [9].

The matrices of interest here are \(n \times n\) truncations of infinite lower triangular (real) matrices, i.e.

\[
A_n = \begin{bmatrix}
a_{1,1} & & \\
a_{2,1} & a_{2,2} & \\
a_{3,1} & a_{3,2} & a_{3,3} & \\
& & & & \ddots & \\
& & & & & a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \\
\end{bmatrix}.
\] (1)

The following result was proven in [11].

\textbf{Theorem 1.1:} Assume that

(i) \(a_{i,j} \geq a > 0, \quad j = 1, \ldots, n, \quad i = j, \ldots, n,\)

\ast Corresponding author. Email: berenhks@wfu.edu
(ii) \( a_{j,j} \geq a_{j+1,j} \geq \cdots \geq a_{n,j}, \quad j = 1, \ldots, n, \)
and let
\[
a_{\min} = \min_{i=1, \ldots, n} \{a_{i,i}\},
\]
(2)
and \( B_n = [b_{i,j}] \) be the inverse of the lower triangular matrix \( A_n \). Then
\[
\|B_n\|_1 \leq \frac{1}{a_{\min}} + \frac{2}{a}.
\]
(3)

The result in (3) was first proven in the case of triangular Toeplitz matrices in [10] and improved, in this case, to the following in [3].

**Theorem 1.2:** Suppose that the sequence \( \{a_i\}_{i \geq 0} \) satisfies
\[
a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq a > 0,
\]
(4)
for some constant \( a \) and all \( n \) and
\[
C_n = \begin{bmatrix}
a_0 & a_1 & a_0 & \cdots & a_n \\
a_1 & a_0 & a_1 & \cdots & a_0 \\
a_2 & a_1 & a_1 & \cdots & a_0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & \cdots & a_1 & a_0
\end{bmatrix}.
\]
(5)
Then
\[
\|C_n^{-1}\|_1 \leq \frac{2}{a} \left( 1 - \rho(a, a_0) \left[ \frac{n}{2} \right] \right)
\]
(6)
where \( \rho \) is the inverse ratio defined via
\[
\rho(x, y) = 1 - x/y,
\]
(7)
and, in particular
\[
\|C_n^{-1}\|_1 \leq \frac{2}{a},
\]
(8)
independent of \( a_0 \) and \( n \).

The following result was recently proven in [2]. The theorem extends Theorem 1.2 (to non-Toeplitz matrices) and refines Theorem 1.1 in the case of constant diagonal.

**Theorem 1.3:** Assume that the hypotheses of Theorem 1.1 are satisfied and in addition that
\[
a_{1,1} \leq a_{2,2} \leq \cdots \leq a_{n,n}.
\]
(9)
Then
\[ \|B_n\|_1 \leq \frac{2}{a} \left( \frac{a_{n,n}}{a_{1,1}} \right) \left( 1 - \frac{\rho(a, a_{n,n})\left(\frac{n}{2}\right) + \rho(a, a_{n,n})\left(\frac{n}{2}\right)}{2} \right). \]  

(10)

In particular, if
\[ a_{1,1} = a_{2,2} = \cdots = a_{n,n} = a^*, \]

then
\[ \|B_n\|_1 \leq \frac{2}{a} \left( 1 - \frac{\rho(a, a^*)\left(\frac{n}{2}\right) + \rho(a, a^*)\left(\frac{n}{2}\right)}{2} \right) \]

(12)

and hence
\[ \|B_n\|_1 < \frac{2}{a}, \]

(13)

independent of \(a^*\).

Note that triangular matrices satisfying (11) arise in the study of linear groups (see for instance [8]) and are particularly important in the theory of matrix decompositions.

Motivated by the above results, here we will improve on Theorem 1.1 by showing that the term \(1/a_{\min}\) in (3) is not needed and in addition that the new bound is in a sense best possible. In particular we will prove the following.

**Theorem 1.4:** Assume that the hypotheses of Theorem 1.1 are satisfied. Then
\[ \|B_n\|_1 \leq \frac{2}{a}. \]

(14)

In fact, setting
\[ A_n(a) = \{ A = [a_{i,j}]_{n \times n} \mid A \text{ satisfies (1), (i) and (ii)} \}, \]

(15)

we have
\[ \sup_{A \in \bigcup_{n \geq 1} A_n(a)} \|A^{-1}\|_1 = 2/a. \]

(16)

The reader is referred to [1] for some discussion of bounds for inverses of matrices of the form in (1) when the condition of monotonicity within columns is replaced with that within rows.
2. Preliminaries and notation

In order to prove Theorem 1.4 we will need several results mentioned in [11]; for completeness we will prove the necessary preliminaries here.

First, define the sequence \( \{U_{i,j}\} \) via
\[
U_{i,j} = \begin{cases} 
0 & \text{if } j > i \\
\frac{1}{a_{j,j}} & \text{if } j = i \\
\frac{1}{a_{i,i}} - \sum_{l=j}^{i-1} \frac{a_{i,l}}{a_{l,l}} U_{l,j} & \text{if } i > j
\end{cases}
\] (17)
Note that for \( i \geq j \), \( U_{i,j} = u_{j,i} \), where \([u_{i,j}]\) is the fundamental matrix as defined in Equation (2.10) in [11].

We have the following (see also Theorem 2.1 in [11]).

Lemma 2.1: Under the assumptions of Theorem 1.1,
\[
U_{m,j} \geq 0,
\] (18)
for all \( m, j \geq 1 \).

**Proof.** For fixed \( j \geq 1 \), we have \( U_{j,j} = 1/a_{j,j} > 0 \). Thus assume the result holds for \( m = j, \ldots, i - 1 \), for some \( i \geq j + 1 \). Then, by Assumption (ii), the induction hypothesis, and the definition in (17),
\[
U_{i,j} = \frac{1}{a_{i,i}} \left( 1 - \sum_{l=j}^{i-1} a_{i,l} U_{l,j} \right) \geq \frac{1}{a_{i,i}} \left( 1 - \sum_{l=j}^{i-1} a_{i-1,l} U_{l,j} \right)
\]
\[
\geq \frac{a_{i-1,i-1}}{a_{i,i}} \left( \frac{1}{a_{i-1,i-1}} - \sum_{l=j}^{i-2} \frac{a_{i-1,l}}{a_{l,l}} U_{l,j} - U_{i-1,j} \right) = 0.
\] (19)
and the result follows. \( \blacksquare \)

Note that in the line corresponding to (19) in the proof of Theorem 2.1 in [11] there is a missing negative sign.

Now, note that the lower triangular matrix \( B_n = [b_{i,j}] = A_n^{-1} \) satisfies \( b_{j,j} = 1/a_{j,j} \) and for \( 1 \leq j < i \leq n \),
\[
b_{i,j} = \sum_{l=j}^{i-1} \frac{a_{i,l}}{a_{i,i}} b_{l,j},
\] (20)
(see for instance [1]).

The next lemma is essentially a restatement of Equation (2.13) in [11].

Lemma 2.2: For all \( 1 \leq j \leq i \leq n \),
\[
b_{i,j} = U_{i,j} - U_{i,j+1}.
\] (21)

**Proof.** For fixed \( j \geq 1 \), we have \( U_{j,j} - U_{j,j+1} = U_{j,j} = 1/a_{j,j} \) and for \( i > j \),
An Optimal Bound for Inverses of Triangular Matrices

\[ U_{i,j} - U_{i,j+1} = \frac{1}{a_{i,i}} - \sum_{l=j}^{i-1} \frac{a_{i,l}}{a_{i,i}} U_{l,j} - \left( \frac{1}{a_{i,i}} - \sum_{l=j+1}^{i-1} \frac{a_{i,l}}{a_{i,i}} U_{l,j+1} \right) \]

\[ = \sum_{l=j}^{i-1} \frac{a_{i,l}}{a_{i,i}} (U_{l,j} - U_{l,j+1}) \quad (22) \]

The last equality in (22) follows since \( U_{j,j+1} = 0 \) (contrast with the definition of \( u_{j,j+1} \) in [11]).

The result follows upon comparing (20) and (22).

3. Proof of the main theorem

We are now in a position to prove Theorem 1.4 (contrast the proof below with that of Theorem 2.3 in [11]).

**Proof of Theorem 1.4.** First note that by employing the definition in (22), (i), (ii) and Lemma 2.1, we have

\[ U_{j,j} \leq \frac{a_{j+1,j+1}}{a_j} \left( \frac{a_{j+1,j}}{a_{j+1,j+1}} U_{j,j} \right) = \frac{a_{j+1,j+1}}{a_j} \left( \frac{1}{a_{j+1,j+1}} - U_{j+1,j} \right) \leq \frac{1}{a}, \quad (23) \]

Similarly

\[ U_{j,j} + U_{j+1,j} \leq \frac{a_{j+2,j+2}}{a_j} \left( \frac{a_{j+2,j}}{a_{j+2,j+2}} U_{j,j} + \frac{a_{j+2,j+1}}{a_{j+2,j}} U_{j+1,j} \right) \]

\[ = \frac{a_{j+2,j+2}}{a_j} \left( \frac{1}{a_{j+2,j}} - U_{j+2,j} \right) \leq \frac{1}{a}, \quad (24) \]

and in general for \( m \geq j \),

\[ \sum_{i=j}^{m} U_{i,j} \leq \frac{a_{m+1,m+1}}{a_m} \left( \sum_{i=j}^{m} \frac{a_{m+1,l}}{a_{m+1,m+1}} U_{l,j} \right) \]

\[ = \frac{a_{m+1,m+1}}{a_m} \left( \frac{1}{a_{m+1,m+1}} - U_{m+1,j} \right) \leq \frac{1}{a}. \quad (25) \]

Employing Lemma 2.2, we have

\[ \|B_n\| = \max_{j=1, \ldots, n} \sum_{i=j}^{n} |b_{i,j}| = \max_{j=1, \ldots, n} \sum_{i=j}^{n} |U_{i,j} - U_{i,j+1}| \]

\[ \leq \max_{j=1, \ldots, n} \left( \sum_{i=j}^{n} U_{i,j} + \sum_{i=j}^{n} U_{i,j+1} \right). \quad (26) \]

Now, noting that \( U_{j,j+1} = 0 \), (25) gives
The bound in (14) then follows upon applying (27) in (26).

Considering the apparent looseness in the inequality in (26), it is perhaps surprising that for large $n$, (14) is in fact optimal. In order to prove (16) we need to show that, in the limit, the bound in (14) is attained. To that end, suppose $a_{i,j} = a^* > 0$ for $i - j \in \{0, 1\}$ and $a_{i,j} \equiv a$ otherwise. It is easy to verify in this case, that for $1 \leq j \leq i \leq n$,

$$b_{i,j} = (-1)^{i-j} \frac{1}{a^*} \left(1 - \frac{a}{a^*}\right) \left\lfloor \frac{i-j}{2} \right\rfloor,$$

and hence,

$$\|A_n^{-1}\|_1 = \sum_{i=1}^{n} |b_{i,1}|$$

$$= \sum_{i=1}^{n} \frac{1}{a^*} \left(1 - \frac{a}{a^*}\right) \left\lfloor \frac{i-j}{2} \right\rfloor = \sum_{i=0}^{n-1} \frac{1}{a^*} \left(1 - \frac{a}{a^*}\right) \left\lfloor \frac{i}{2} \right\rfloor$$

$$= \frac{1}{a^*} \left( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \left(1 - \frac{a}{a^*}\right)^i + \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \left(1 - \frac{a}{a^*}\right)^i \right)$$

$$= \frac{2}{a} \left( 1 - \rho(a, a^*) \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} + \rho(a, a^*) \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right).$$

The result follows upon letting $n$ tend to infinity in (29).  

References

Chapter 5: Conclusion and Future Research

This thesis has examined solutions to difference equations of convolution type and their related applications in matrix analysis, and has developed some reasonably broad results. We used local properties of \( \{b_n\} \) to extend work of Elaydi which had considered \( \{b_n\} \) only in the limit. But as demonstrated in Example 1 of Chapter 2, local consideration of \( \{b_n\} \) is absolutely crucial if we wish to have any real understanding of the sequence’s impact on \( \{x_n\} \).

It should be noted that all our results use a specific local property of \( \{b_n\} \), namely monotonicity. Considering this, it would be a logical next step to extend our consideration to sequences of monotone type, including quasi-monotone sequences. There are often ready extensions of monotone results in the study of quasi-monotone sequences, but it remains to be seen if our techniques would find traction. Looking beyond quasi-monotone sequences, any family of sequences in which local behavior is governed by a global restriction presents an opportunity for further investigation.

Of course, future developments in any of these cases may be used to consider bounds on matrices. At present, working from the foundation of difference equations seems to be an effective means of obtaining bounds for matrix inverses.
Bibliography


Vita

Nathaniel G. Vish

78 South Main St. Tel. 828.406.1954
Weaverville, NC 28787 E-mail: nathanvish@gmail.com

Educational Record

• MA, Mathematics, Wake Forest University (anticipated graduation 2009)
  – Advisor: Dr. Kenneth S. Berenhaut
  – Thesis: “Some New Results on Difference Equations of Convolution Type”

• BS, Mathematics with English concentration, Appalachian State University (2007)
  – Advisor: Dr. Jeffry Hirst
  – Thesis: “Scaled Structures in Morse-Thue and Related Sequences”

• AS, Mathematics Premajor, Asheville-Buncombe Technical Community College (2005)
  – Dual-enrolled student, August 2001 - May 2003
  – Full-time student, August 2004 - May 2005
  – Advisor: R. Trent Codd, Jr.

• Non-degree seeking student, University of North Carolina Asheville (2003)
  – Dual-enrolled student, Analytical Physics II, spring 2003

Professional Experience

• Teaching Assistant, Wake Forest University (2007 - 2009)
• Private Tutor, Mathematics (2002 - 2009)
• Teaching Assistant, Cryptology, Johns Hopkins Center for Talented Youth (sessions I - II, summer 2005 - summer 2007)

• Volunteer Tutor, Mathematics, Appalachian State University (2005 - 2006)

• Peer Tutor, Mathematics, Chemistry, Physics, Asheville-Buncombe Technical Community College (2002 - 2005)

Meetings and Conferences

• Presenter, AMS/MAA Joint Sessions, Washington, DC (January 2009)
  – Contributed talk, “Equations of Convolution Type with Monotone Coefficients”

• Presenter, North Carolina Mini-Conference on Graph Theory, Combinatorics, and Computing, Appachian State University, Boone, NC (April 2007)
  – Contributed undergraduate talk, “Scaled Structures and Self-Similarity in the Morse-Thue Sequence”

• Presenter, MAA–Southeastern Sectional, GSU, Statesboro, GA (March 2007)
  – Contributed undergraduate talk, “Scaled Structures and Self-Similarity in the Morse-Thue Sequence”; recipient of the Patterson Prize for outstanding undergraduate research presentation

• Presenter, AMS/MAA Joint Sessions, New Orleans, LA (January 2007)
  – Poster presentation of original research in combinatorics on words, “Scaled Structures and Self-Similarity in the Morse-Thue Sequence”

• Presenter, State of North Carolina Undergraduate Research and Creativity Symposium, NCSU, Raleigh, NC (November 2006)
  – Poster presentation of original research in combinatorics on words, “Scaled Structures and Self-Similarity in the Morse-Thue Sequence”

Scholarships and Grants

• Wake Forest University Chambers Grant for eCommerce and Internet Startups, PR and marketing internship with Involve: A Journal of Mathematics, $1500 (2008)

• NSF-CSEMS scholarship for mathematics, $14,000 (2005 - 2007)
• Appalachian State University Office of Student Research travel grant (2007)
• Appalachian State University Mathematical Honors program participant (2005 - 2007)
• Virginia Tech Undergraduate Research Workshop funded participant (2006)

Publications

• *Equations of convolution type with monotone coefficients*, Journal of Difference Equations and Applications (revised manuscript in review for publication)
• *An optimal bound for inverses of triangular matrices with monotone entries*, Linear and Multilinear Algebra (revised manuscript in review for publication)
• *A 1-norm bound for inverses of triangular matrices with monotone entries*, Banach Journal of Mathematical Analysis (Volume 2, Number 1, 2008)
• *Scaled Structures and Self-Similarity in the Morse-Thue Sequence*, Explorations—the North Carolina journal of undergraduate research (Volume 2007)

Awards

• Runner-up winner, 9th Annual Wake Forest Graduate Student Research Day poster presentation, “Equations of Convolution Type with Monotone Coefficients,” Wake Forest University (2009)
• Outstanding Senior award, Department of Mathematical Sciences, Appalachian State University (2007)
• Academic Honor Student, Appalachian State University (2007)
• Patterson Prize for outstanding undergraduate research presentation, “Scaled Structures and Self-Similarity in the Morse-Thue Sequence,” contributed talk, MAA-Southeastern Sectional, GSU (2007)
• Pi Mu Epsilon National Mathematics Honors Society, Wake Forest University
• Phi Eta Sigma National Honors Society, Appalachian State University
• Phi Theta Kappa, Alpha Upsilon Eta Chapter, Asheville Buncombe Technical Community College
• Chancellor’s List, Appalachian State University (2005 - 2007)
• Dean’s List, Appalachian State University (2005 - 2007)
• President’s List, Asheville-Buncombe Technical Community College (2003 - 2005)