THE OCCURRENCE OF FIBONACCI
AND LUCAS NUMBERS IN THE
GEOMETRY OF $\mathcal{H}(\mathbb{R}^N)$

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1 INTRODUCTION

The set of all non-empty compact subsets of $\mathbb{R}^N$ under the Hausdorff metric, $h$, forms a complete metric space, which we will denote as $\mathcal{H}(\mathbb{R}^N)$. For certain types of elements $A$ and $B$ in $\mathcal{H}(\mathbb{R}^N)$, we will show that the number of elements between $A$ and $B$ at each location is exactly a Fibonacci or Lucas number.

2 THE HAUSDORFF METRIC

A metric is a function which measures distance on a space. We will denote the standard Euclidean distance between $x$ and $y$ in $\mathbb{R}^N$ as $d_E(x, y)$. The Hausdorff metric, defined below, imposes a geometry on the space $\mathcal{H}(\mathbb{R}^N)$, which will be the subject of our study. To distinguish between $\mathbb{R}^N$ and $\mathcal{H}(\mathbb{R}^N)$, we will refer to points in $\mathbb{R}^N$ and elements in $\mathcal{H}(\mathbb{R}^N)$.

Definition 2.1. Let $A$ and $B$ be elements in $\mathcal{H}(\mathbb{R}^N)$.

- If $x \in \mathbb{R}^N$, the “distance” from $x$ to $B$ is
  $$d(x, B) = \min_{b \in B} \{d_E(x, b)\}.$$  
  A picture of this can be seen in Figure 1.

- The “distance” from $A$ to $B$ is
  $$d(A, B) = \max_{x \in A} \{d(x, B)\}.$$  
  Note that $d$ is not a metric, since $d$ is not symmetric. That is, $d(A, B) \neq d(B, A)$ in general, as shown in Figure 2.

- The Hausdorff distance, $h(A, B)$, between $A$ and $B$ is
  $$h(A, B) = d(A, B) \vee d(B, A),$$  
  where $d(A, B) \vee d(B, A) = \max\{d(A, B), d(B, A)\}$. 
  See [1] for a proof that $h$ is a metric on $\mathcal{H}(\mathbb{R}^N)$. 

Figure 1: Distance from a point to an element.
Thus, \( A + M \) for all \( \{ A \} \).

Proof. First we show that \( A + M \) for all \( \{ A \} \). Moreover, if \( C \) is in \( \{ A \} \). Let \( S(\{ A \}, B) \) and \( S(\{ A \}, B) \) be Hausdorff segments. We will denote the set of elements \( C \in \{ A \} \) that lie between \( A \) and \( B \) as \( S(\{ A \}, B) \) and call this set the Hausdorff segment defined by \( A \) and \( B \).

An interesting property of Hausdorff segments is possibility for the presence of more than one distinct element at a specific location between the defining elements. The following definition describes what is meant when we refer to two elements at the same location on a Hausdorff segment.

**Definition 3.2.** Let \( A \neq B \in \{ A \} \) and \( C, C' \in S(\{ A \}, B) \). The elements \( C, C' \) are said to be at the same location between \( A \) and \( B \) if \( h(B, C) = h(B, C') = s \) for some \( 0 < s < h(A, B) \).

### 3.1 Hausdorff Segments

In this section we define segments in \( \{ A \} \). It is in this context that we will later encounter the Fibonacci and Lucas numbers.

In the standard Euclidean geometry, when a point \( x \) lies between the points \( a \) and \( b \), we have \( d_E(a, b) = d_E(a, x) + d_E(x, b) \). Since this concept only depends on the distance between two points, we can apply it to elements in \( \{ A \} \).

**Definition 3.1.** Let \( A \neq B \in \{ A \} \). The Hausdorff segment defined by \( A \) and \( B \) is the set of all elements \( C \in \{ A \} \) for which \( h(A, B) = h(A, C) + h(C, B) \).

Notation: If \( h(A, B) = h(A, C) + h(C, B) \), we say that \( C \) is between \( A \) and \( B \), and denote this by \( ACB \) as in [2]. We will denote the set of elements \( C \in \{ A \} \) that lie between \( A \) and \( B \) as \( S(\{ A \}, B) \) and call this set the Hausdorff segment defined by \( A \) and \( B \).

3.2 **Extensions**

One construction that will play a crucial role in the appearence of Fibonacci and Lucas numbers in the geometry of \( \{ A \} \) is the extension of a set. Given an element \( B \in \{ A \} \) (that is, a non-empty compact subset of \( \{ A \} \) and \( r > 0 \), the extension of \( B \) by \( r \) is the set \( B + r = \{ x \in \{ A \} : d(x, b) \leq r \) for some \( b \in B \} \).

An important and useful result about extensions is the following.

**Proposition 3.1.** Let \( A \in \{ A \} \) and let \( s > 0 \). Then \( A + s \) is a compact set that is a distance \( s \) from \( A \). Moreover, if \( C \in \{ A \} \) and \( h(A, C) \leq s \), then \( C \subseteq A + s \).

**Proof.** First we show that \( A + s \) is in \( \{ A \} \).

Since \( A \in \{ A \} \), we know that \( A \) is closed and bounded. Let \( M \) be a bound for \( A \). Thus, \( |a| = d(a, 0) \leq M \) for all \( a \in A \). Let \( x \in A + s \). Then there is an element \( a_x \in A \) so that \( d(x, a_x) \leq s \). So

\[
d(x, 0) \leq d(x, a_x) + d(a_x, 0) \leq s + M.
\]

Thus, \( A + s \) is bounded by \( M + s \).

Next we demonstrate that \( A + s \) is closed. Let \( x \) be a limit point of \( A + s \). So there is a sequence \( \{ x_m \} \) in \( A + s \) that converges to \( x \). For each \( x_m \) there is a point \( a_m \in A \) so that \( d(x_m, a_m) \leq s \). Since the sequence \( \{ a_m \} \) is a bounded sequence (\( A \) is bounded), \( \{ a_m \} \) has a convergent subsequence \( \{ b_l \} \). Since \( A \) is closed,
\(a = \lim b_t\) is an element of \(A\). Now, let \(\epsilon > 0\) and choose \(K\) so that \(m, l > K\) implies \(d(x, x_m) < \frac{\epsilon}{2}\) and \(d(a, b_t) < \frac{\epsilon}{2}\). Then for \(t > K\), we have

\[
d(x, a) \leq d(x, x_t) + d(x_t, b_t) + d(b_t, a) < \epsilon + s.
\]

This shows that \(d(x, a) \leq s\) and \(x \in A + s\). Therefore, \(A + s\) is compact.

Now we show that \(h(A, A + s) = s\). Note that \(A \subseteq A + s\), so \(d(A, A + s) = 0\). For each \(x \in A + s\), there is an \(a_x \in A\) so that \(d(x, A) \leq d(x, a_x) \leq s\). Therefore, \(d(A, s, A) \leq s\). To obtain equality, we only need to find an element in \(A + s\) not in \(A\) that is a distance \(s\) from some element in \(A\). Let \(a \in A\) with \(|a|\) a maximum. Let \(x = (1 + s)a\). In other words, \(x\) is the point on the line through the origin and \(a\) that is \(s\) units farther from the origin than \(a\). So \(x \in A + s\). Clearly, \(|x| > |a|\), so \(x \not\in A\). Since \(|a|\) is maximum over all elements in \(A\), we also have \(d(x, a') \leq d(x, a)\) for all \(a' \in A\). Therefore, \(d(x, A) = d(x, a) = s\) and \(h(A, A + s) = s\).

Finally, we show that \(A + s\) is the largest element of \(\mathcal{H}(\mathbb{R}^N)\) that is a Hausdorff distance \(s\) from \(A\). Let \(C \in \mathcal{H}(\mathbb{R}^N)\) with \(h(A, C) = s\). Let \(c \in C\). The fact that \(h(A, C) = s\) implies that there is an element \(a \in A\) so that \(d(a, c) \leq s\). Therefore, \(c \in A + s\). So \(C \subseteq A + s\).

### 3.3 Finding Points on a Hausdorff Segment

Let \(A \neq B \in \mathcal{H}(\mathbb{R}^N)\). In this section we will see that Hausdorff segments fall into two categories: those containing an infinite number of elements at each location (except at the locations of either \(A\) or \(B\)) and those containing a finite number of elements at each location. In [5], the author proves the following lemma (where the author uses the alternate notation \((A)_s\) to mean the extension of \(A\) by \(s\)).

**Lemma 3.1.** Let \(A, B \in \mathcal{H}(\mathbb{R}^N)\), \(h(A, B) = \alpha\) and let

\[
M(s) = (A)_s \cap (B)_{(\alpha - s)}
\]

for each \(s \in [0, \alpha]\). Then \(h(A, M(s)) = s\) and \(h(M(s), B) = \alpha - s\).

Bay and Schlicker [4] extended the previous result to find more elements on Hausdorff segments.

**Theorem 3.1.** Let \(A \neq B \in \mathcal{H}(\mathbb{R}^N)\) with \(d(B, A) \geq d(A, B)\). Let \(r = h(A, B)\). Let \(s \in \mathbb{R}\) with \(0 < s < r\), and let \(t = r - s\). If \(C\) is a compact subset of \((A + s) \cap (B + t)\) containing \(\partial((A + s) \cap (B + t))\), then \(C\) satisfies \(ACB\) with \(h(A, C) = s\) and \(h(B, C) = t\).

Theorem 3.1 tells us that if \((A + s) \cap (B + t)\) has an infinite interior, then there will be an infinite number of elements in \(\mathcal{H}(\mathbb{R}^N)\) at each location between \(A\) and \(B\). An example of this situation occurs when \(A = \{0\}\) and \(B = \{1, 2\}\) in \(\mathcal{H}(\mathbb{R})\). In this case, \(h(A, B) = d_E(0, 2) = 2\). Note that for any \(0 < s < 2\), we have \((A + s) \cap (B + t) = [-s, s]\). If \(C\) is a compact subset of \([-s, s]\) containing \([-s, s]\), then \(h(A, C) = s\) and \(h(B, C) = 2 - s\). So we have an infinite number of elements between \(A\) and \(B\) that are \(s\) units from \(A\).

Alternatively, if \((A + s) \cap (B + t)\) is finite, Proposition 3.1 shows us that an element \(C \in \mathcal{H}(\mathbb{R}^N)\) with \(h(A, C) = s\) and \(h(B, C) = t\) must be a subset of both \(A + s\) and \(B + t\). In this situation we can have at most a finite number of elements at each location between \(A\) and \(B\). An example of this occurs when \(A\) and \(B\) are both single points sets. In this case, \((A + s) \cap (B + t)\) will always be a single point set for \(0 < s < h(A, B)\).

A more interesting example of a segment containing a finite number of elements at each location is the following.

**Example 3.1.** Let \(A = \{0.5, 0.5\}, \{0.5, -0.5\}, \{-0.5, 0.5\}, \{-0.5, -0.5\}\) and \(B = \{2, 0\}, \{0, 2\}, \{-2, 0\}, \{0, -2\}\) in \(\mathcal{H}(\mathbb{R}^2)\). We see that if \(s, t \in \mathbb{R}^+\) with \(r = s + t\), then \(C = (A + s) \cap (B + t)\) is an eight-point set that lies on the Hausdorff segment \(S(A, B)\) as illustrated in Figure 3. In fact, \(C\) is only one of 47 elements at each location on \(S(A, B)\). Interestingly, 47 is the eighth Lucas number, \(L_8\). In the next section, we will present several more general examples of the occurrence of the Lucas and Fibonacci numbers in the geometry of \(\mathcal{H}(\mathbb{R}^N)\).

For now, we will find all 47 elements that lie on the Hausdorff segment between \(A\) and \(B\). To begin, we note that the largest element between \(A\) and \(B\) is \(C = \{1, 2, 3, 4, 5, 6, 7, 8\}\) where the labeling system is shown in Figure 3. To find the other 46 elements, we will be taking certain subsets of \(C\) as follows:

1. \(C - \{a\}\) where \(a \in C\) (There will be 8 elements)
2. \(C - \{a, b\}\) where \(a \neq b \in C\) and \(a\) and \(b\) are not adjacent to each other (There will be 20 elements)
3. $C = \{a, b, c\}$ where $a \neq b \neq c \in C$ and $a$ is not adjacent to $b$ or $c$ and $b$ is not adjacent to $c$ (There will be 16 elements)

4. $C = \{1, 3, 5, 7\}$ and $C = \{2, 4, 6, 8\}$

It is left as an exercise for the reader to verify that each of these sets lies at the same location as $C$ on $S(A, B)$. Thus we have found all 47 elements between $A$ and $B$ at a given location.

We have seen examples of Hausdorff segments with an infinite number of elements at each location as well as segments with a finite number of elements at each location. One question which comes to mind is, how can we determine which Hausdorff segments will have a finite number of elements at each location? The complete answer to that question is not within the scope of this paper, but there are some useful basic characteristics of elements which define Hausdorff segments with a finite number of elements at each location. Complete details of the proofs of Theorems 3.2 and 3.3 can be found in [8].

In order for a pair of elements $A$ and $B$ in $H(\mathbb{R}^n)$ to define a Hausdorff segment with a finite number of elements at each location, every point in $A$ must be a common distance $r = h(A, B)$ away from $B$ and every point in $B$ must be $r$ units away from $A$. If this condition is not met, the Hausdorff segment $S(A, B)$ will have an infinite number of elements at each location. This fact is summarized in Theorem 3.2.

**Theorem 3.2.** For $A$ and $B$ in $H(\mathbb{R}^n)$ with $h(A, B) = r$, if there exists a point $b \in B$ such that $d(b, A) \neq r$, then there are infinitely many elements at a given location on the Hausdorff segment between $A$ and $B$.

The reader might ask, can there be $n$ elements at one location on a Hausdorff segment and $m \neq n$ elements at another? It is the case, however, that there are exactly the same number of elements at every location between $A$ and $B$ on $S(A, B)$. At an endelement (e.g. the location of element $A$), there is only one element on the Hausdorff segment $S(A, B)$. These facts are stated in Theorem 3.3.

**Theorem 3.3.** Let $A, B$ be elements in $H(\mathbb{R}^n)$ which contain a finite number of points in $\mathbb{R}^N$. If all points $b \in B$ are equidistant from $A$ and all points $a \in A$ are equidistant from $B$, then there is the same finite number of elements at every location between $A$ and $B$ on the Hausdorff segment $S(A, B)$, with the possible exception of the locations of the endelements.

### 4 FIBONACCI AND LUCAS NUMBERS IN THE GEOMETRY OF $H(\mathbb{R}^N)$

We now turn to some specific examples of Hausdorff line segments between elements $A, B \in H(\mathbb{R}^n)$ that yield Fibonacci and Lucas numbers.

#### 4.1 HAUSDORFF STRING SEGMENTS

Let $r > 0$ and let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_m\}$, where $a_i$ is the point $(2i - 2)r$ units along the positive $x$-axis, $b_j$ is the point $(2j - 1)r$ units along the positive $x$-axis, $n = k + m$, and either $k = m$
or \( k = m + 1 \). Note that \( d(a_i, B) = r = d(b_j, A) \) for each \( i \) and \( j \). Let \( 0 < s < r \) and \( t = r - s \). We will determine the number of elements \( C \) in \( \mathcal{H}(\mathbb{R}^n) \) satisfying \( ACB \) with \( h(A, C) = s \). Proposition 3.1 tells us that \( C \) will be a subset of \( (A + s) \cap (B + t) \).

Let \( F_{n-1} \) denote the number of elements \( C \) satisfying \( ACB \) with \( h(A, C) = s \). Recall, due to Proposition 3.3, that \( F_{n-1} \) will retain the same finite value for all locations on \( S(A, B) \). We begin our investigation with a few special cases.

I. The simplest case occurs when \( k = m = 1 \) (i.e. when \( A \) and \( B \) are singleton sets), which was considered in [3]. In this case we have \( F_1 = 1 \).

II. Now suppose \( A = \{a_1, a_2\} \) and \( B = \{b_1\} \). Note that \( (A + s) \cap (B + t) \) is a two point set \( X = \{x_1, x_2\} \), with \( x_1 < x_2 \). Each set \( C \) in question will be a subset of \( X \). If \( x_1 \notin C \), then \( h(A, C) = d(a_1, x_2) > s \). A similar argument shows that \( C \) contains \( x_2 \). Thus, \( C = X \) and \( F_2 = 1 \).

III. Consider \( A = \{a_1, a_2\} \) and \( B = \{b_1, b_2\} \). Note that \( (A + s) \cap (B + t) \) is a three point set \( X = \{x_1, x_2, x_3\} \), with \( x_1 < x_2 < x_3 \) (see the figure above). Again, each set \( C \) in question will be a subset of \( X \). As above, if \( x_1 \notin C \), then \( h(A, C) = d(a_1, x_2) > s \). A similar argument shows that \( C \) contains \( x_3 \). Notice that both \( C = X \) and \( C = \{x_1, x_3\} \) satisfy \( ACB \) with \( h(A, C) = s \). Therefore, \( F_3 = 2 \).

You might see a pattern emerging. The next theorem makes this specific.

**Theorem 4.1.** \( F_n \) is the \( n \textsuperscript{th} \) Fibonacci number.

The proof of Theorem 4.1 is straightforward and the method is probably familiar to readers of this journal. Let \( A = \{a_1, a_2, \ldots, a_k\} \) and \( B = \{b_1, b_2, \ldots, b_m\} \) with \( n = k + m \). Let \( s \) be a number between 0 and \( r, t = r - s \). The set \( (A + s) \cap (B + t) \) is an \( n - 1 \) point set \( X = \{x_1, x_2, x_3, \ldots, x_{n-1}\} \), with \( x_1 < x_2 < x_3 < \cdots < x_{n-1} \). Each set \( C \) satisfying \( ACB \) with \( h(A, C) = s \) and \( h(B, C) = t \) will be a subset of \( X \). The points \( x_1 \) and \( x_{n-1} \) must be in \( C \). There are two cases to consider: \( x_2 \in C \) and \( x_2 \notin C \). The number of elements between \( A \) and \( B \) containing \( x_2 \) is the same as the number of elements between \( B \) and \( A' \) = \{\( a_2, \ldots, a_k\)\}. By induction, this is \( F_{n-1} \). If \( x_2 \notin C \), then \( x_3 \) must be in \( C \). Then the number of elements between \( A \) and \( B \) not containing \( x_2 \) is the same as the number of elements between \( A' \) and \( B' \) = \{\( b_2, \ldots, b_m\)\}. By induction, this is \( F_{n-2} \). Therefore,

\[
F_n = F_{n-1} + F_{n-2}.
\]

The formal details are left to the reader. \( \square \)

So the Fibonacci numbers occur naturally as the number of elements at each location on a Hausdorff String segment. One interesting consequence of this is that all Hausdorff segments in \( \mathcal{H}(\mathbb{R}) \) that contain only a finite number of elements at each location must be constructed out of String segments and therefore have a product of Fibonacci numbers as the number of elements at each location between the endelements. To see how Lucas numbers arise in Hausdorff segments, we must move to \( \mathcal{H}(\mathbb{R}^2) \).

### 4.2 HAUSDORFF POLYGONAL SEGMENTS

Let \( A = \{a_1, a_2, a_3, a_4\} \) and \( B = \{b_1, b_2, b_3, b_4\} \) each be the set of vertices of a single square, as seen in Figure 5. We see that the distance \( d(a, B) = d(h, A) \) for all \( a \) in \( A \) and all \( b \) in \( B \). This configuration is analogous to...
the one we saw in Example 3.1. In fact, there are 47 sets that lie at each location on the Hausdorff segment between $A$ and $B$, all such sets are exhaustively listed in Example 3.1.

The general construction is as follows. Let $A$ and $B$ be vertices of regular $n$-gons with $n \in \mathbb{N}$ in which the $n$-gons share the same center point and initially are stacked such that the vertices correspond. Then $B$ is rotated $\frac{\pi}{n}$ radians with respect to $A$ about the center point. By Proposition 3.3, the Hausdorff segment between $A$ and $B$ will have a finite number of elements at each location between $A$ and $B$. We now show that the precise number of elements at each location, denoted by $p(n)$, can be calculated using either of the following two formulas, where $L_n$ denotes the $n^{th}$ Lucas number:

$$p(n) = F_{2n} + 2 \cdot F_{2n-1} = L_{2n}$$  \hspace{1cm} (4.1)$$

or

$$p(n) = \sum_{k=0}^{n} \binom{2n-k+1}{k} - \sum_{j=0}^{n} \binom{2n-4-j+1}{j}.$$  \hspace{1cm} (4.2)$$

The derivation of (4.1) is much like that of Theorem 4.1. We will focus on 4.2. This formula follows from Lucas’ result, $F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \ldots + \binom{n-j-1}{j}$ where $j = [(n-1)/2]$ (see [7]). An outline of an inductive proof of this result can be found on page 274 in [6].

Let $n \in \mathbb{N}$ and let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where $a_1$ is selected from the vertices of one $n$-gon and the point $a_i$ is the $i+1^{st}$ vertex from $a_1$ moving clockwise on the same $n$-gon. On the second $n$-gon, which was rotated $\frac{\pi}{n}$ about the center, $b_1$ is the first vertex that lies $\frac{\pi}{n}$ degrees clockwise from $a_1$ and the point $b_j$ is the $j+1^{st}$ vertex moving clockwise from $b_1$ on the same $n$-gon. Now let $d(a_i, B) = r = d(b_j, A)$ for each $i$ and let $0 < s < r$ and $t = r - s$. Let $C = (A + s) \cap (B + t)$. Then $C$ is on the Hausdorff segment $S(A, B)$.

Now, $C$ is the set of vertices of a regular $2n$-gon. We first choose one of the elements of $C$ and number it 1. We then proceed to number the remaining elements, up to $2n$, moving clockwise around the $2n$-gon. Notice that any subset of $C$ with no two consecutive elements omitted (Note: 1 and $2n$ are consecutive) remains distance $s$ from $B$ and distance $t$ from $A$, and therefore at the desired location on the segment $S(A, B)$. This is not the case when two consecutive elements are removed. Hence, the number of elements at each distance $s$ from $B$ on the Hausdorff segment $S(A, B)$ is equal to the number of possible ways to select any number of objects from a string of $2n$ objects without selecting two consecutive objects (again, with 1 and $2n$ consecutive).

If we wanted to select $k$ objects from a $2n$ object string (this time with the 1 and $2n$ not consecutive) without selecting two consecutive objects, we could count the number of ways to do so in the following manner:

Let there be $2n - k$ white pencils (representing the unselected objects) and $k$ black pencils (representing the selected objects). Arrange the white pencils in a row. Note that there are $2n - k + 1$ spaces between the white pencils, including the space before the first and the space after the last. Now, we select $k$ of these
spaces and insert the black pencils into the selected spaces (one per selected space) to create a string of $2n$ pencils with no two black pencils consecutive. Thus, the number of ways to insert the black pencils is also the number of ways to select $k$ objects from the $2n$ such that no two consecutive objects are chosen. There are $\binom{2n-k+1}{k}$ ways to accomplish this. Summing the ways to select $k$ objects in such a manner for all $k \leq n$, we see that there are $\sum_{k=0}^{n} \binom{2n-k+1}{k}$ ways to select objects from a $2n$ string without selecting two consecutive objects. Note that if $k > n$ then two of the $k$ selected objects must be consecutive. Therefore the cases where $k > n$ do not add to our sum.

Now, we find the number of ways to select $k$ pairwise non-consecutive objects with 1 and $2n$ considered consecutive. Given the above formula for when 1 and $2n$ are not consecutive, we need only subtract the cases where, in the above process, we selected both the 1st and $2n$th objects for removal. The resulting difference should then count exactly the number of ways to remove any number of objects from a $2n$ closed string without removing any consecutive objects. To count the number of cases where both the 1st and $2n$th objects were selected for removal, we fix the 1st and $2n$th object as being selected. This requires the second and $2n-1$th objects to remain unselected. Otherwise, we would have consecutive objects selected in our open $2n$-object string. However, the remaining $2n-4$ “interior” objects can be selected for removal in any way such that there are no consecutive objects selected. But we already have a formula for the number of ways to select $j$ non-consecutive objects of an open $2n-4$ object string: $\binom{2n-4-j+1}{j}$.

Therefore, our formula $p(n) = \sum_{k=0}^{n} \binom{2n-k+1}{k} - \sum_{j=0}^{n} \binom{2n-4-j+1}{j}$ does in fact count the number of ways to select any number of non-consecutive objects from a $2n$ string, which is also the number of elements at any location on the segment $S(A, B)$.

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References

A Reviewer’s Appendix of Complete Proofs

The following Lemma will be useful in the proof of Theorem 3.2.

**Lemma A.1.** Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ and $h(A, B) = r$. Also, Let $0 < s < r$ and $t = r - s$. If two points, $a_0 \in A$ and $b_0 \in B$, exist such that $d(a_0, b_0) < r$, then $N_s(b_0) \cap N_t(a_0) \neq \emptyset$.

**Proof.** Let $l(v) = \left(\frac{v}{d(a_0, b_0)}\right) : b_0 + \left(\frac{d(a_0, b_0) - v}{d(a_0, b_0)}\right) : a_0$ where $v \in [0, d(a_0, b_0)]$ be the Euclidean line segment $\overline{a_0b_0}$. Note that this is an arc-length parameterization of the segment with $l(0) = b_0$ and $l(d(a_0, b_0)) = a_0$. That is, if $u, w \in [0, d(a_0, b_0)]$, then $|l(u) - l(w)| = |u - w|$.

Now, if $d(a_0, b_0) \leq s$, then $a_0$ and the open interval $(l(0), l(q)) \subseteq \overline{a_0b_0}$, with $q = \min\{t, d(a_0, b_0)\}$, is contained in $N_s(b_0) \cap N_t(a_0)$. A similar open interval of $\overline{a_0b_0}$ containing $b_0$ is in $N_s(b_0) \cap N_t(a_0)$ if $d(a_0, b_0) \leq t$. Therefore, we only need look at the case when $d(a_0, b_0) > s$ and $d(a_0, b_0) > t$.

In this case, let $0 < q < r - d(a_0, b_0) < r - t = s$ and consider the point $c = l(s - q)$. Since, $(s - q - 0) < (s - 0)$ and $l$ is an arc-length parameterization, $d(b_0, c) = |l(s - q) - l(0)| = |l(s) - l(0)| = s$. Thus, $c \in N_s(b_0)$.

Now, since $d(a_0, c) + d(c, b_0) = d(a_0, b_0)$ and $d(b_0, c) = s - q$, by substitution $d(a_0, c) = d(a_0, b_0) - s + q$. By substitution, $d(a_0, c) < (r - d(a_0, b_0)) = r - s = t$. Thus, $c \in N_t(a_0)$.

Therefore, $c \in N_s(b_0) \cap N_t(a_0)$ and our lemma is proved.

**Complete Proof of Theorem 3.2:**

**Proof.** Let $A$ and $B$ be in $\mathcal{H}(\mathbb{R}^n)$ with $h(A, B) = r$. Assume there exists a point $b_0 \in B$ such that $d(b_0, A) \neq r$. This implies that $d(b_0, A) < r$, by Definition 2.1. By definition, there exists a point $a_0 \in A$ such that $d(b_0, a_0) = d(b_0, A)$. Let $0 < s < r$ and $t = r - s$. By Lemma A.1, there exists a point $c \in N_s(b_0) \cap N_t(a_0)$.

Now, because $N_s(b_0) \cap N_t(a_0)$ is an open set, there exists a $\delta > 0$ such that $N_\delta(c) \subset (N_s(b_0) \cap N_t(a_0)) \subset ((B + s) \cap (A + t))$.

By Theorem 3.1, we can then choose any compact subset $V$ of $N_\delta(c)$ and $V \cup \partial((B + s) \cap (A + t))$ will satisfy the conditions to be between $A$ and $B$. Therefore there are infinitely many sets at each location on the Hausdorff segment between $A$ and $B$.

**Complete Proof of Theorem 3.3:**

**Proof.** Let $A, B$ be elements in $\mathcal{H}(\mathbb{R}^n)$ such that $|A| = m$ and $|B| = n$ for some $m, n \in \mathbb{N}$. Further, let $d(a, B) = d(b, A) = h(A, B) = r$ for all $a \in A$ and all $b \in B$. Finally, let $0 < s < r$ and $t = r - s$.

We want to show that there are a finite number of elements at every location $s$ away from $B$ on the Hausdorff segment $S(A, B)$. We claim that it is sufficient to prove that $|N_s(b_0) \cap (A + t)| \leq m$ for all $b_0 \in B$.

For if $|N_s(b_0) \cap (A + t)| \leq m$ for all $b_0 \in B$, then $|(B + s) \cap (A + t)| \leq n \cdot m$. Now, any Point $C$ which lies on the Hausdorff segment $S(A, B)$ must be some collection of the points contained in $(B + s) \cap (A + t)$.

Therefore, an upper bound on the number of elements at each location between $A$ and $B$ is the number of ways to choose fewer than $n \cdot m$ elements from a set of $n \cdot m$ elements, or $\sum_{i=0}^{n \cdot m} \binom{n \cdot m}{i}$. But, this is a finite sum, and so must be less than some $N \in \mathbb{N}$. Thus, the number of elements at each location on the segment between $A$ and $B$ is finite.

We now prove that $|N_s(b_0) \cap (A + t)| \leq m$, for all $b_0 \in B$.

Let $b_0 \in B$. Then $d(b_0, a) \geq r$ for all $a \in A$, because $r = \min\{d(b_0, a) : a \in A\}$. We construct a set $A'_0 = \{a \in A : d(b_0, a) = r\}$. The reason for this construction is that points in $A$ for which $d(b_0, a) > r$ do not contribute points to the intersection $N_s(b_0) \cap (A + t)$. That is, $N_s(b_0) \cap N_t(a) = \emptyset$ for all $a \in A \setminus A'_0$. Notice that $1 \leq |A'_0| \leq m$.

Let $a_0 \in A'_0$. We want to show that $N_s(b_0) \cap N_t(a_0) = \{c_0\}$ for some $c_0 \in \mathbb{R}^n$.

First, we show that such a $c_0$ exists. Let $\overline{a_0b_0}$ denote the Euclidean line segment from $a_0$ to $b_0$. Then, from basic Euclidean geometry, there exists exactly one point $c_0 \in \overline{a_0b_0}$ with $d(a_0, c_0) = t$ and $d(c_0, b_0) = s$.

By definition, $c_0$ is an element of $N_s(b_0) \cap N_t(a_0)$.

Next, we claim that no other point $c_0$ exists in $\mathbb{R}^n$ such that $c_0 \in N_s(b_0) \cap N_t(a_0)$.
By contradiction, suppose that such an \( e_0 \) exists. Then \( d(a_0, e_0) \leq t \) and \( d(e_0, b_0) \leq s \), implying that 
\[ d(a_0, e_0) + d(e_0, b_0) \leq s + t = r. \]
Also, we know that \( r = d(a_0, b_0) \leq d(a_0, e_0) + d(b_0, e_0) \) by the Euclidean triangle inequality. Thus \( d(a_0, e_0) = t \), \( d(b_0, e_0) = s \), and \( e_0 \in a_0b_0 \). But since there is only one such point on \( a_0b_0 \), it must be that \( e_0 = c_0 \).

Now, we know that
\[
N_s(b_0) \cap (A + t) = N_s(b_0) \cap (A_0' + t) = N_s(b_0) \cap \left( \bigcup_{a_0 \in A_0'} N_t(a_0) \right)
\]
We also know that, for every \( a_0 \in A_0' \), exactly one point will be added to \( N_s(b_0) \cap \left( \bigcup_{a_0 \in A_0'} N_t(a_0) \right) \). Since there are a maximum of \( m \) points in \( A_0' \), there can be at most \( m \) points in \( N_s(b_0) \cap \left( \bigcup_{a_0 \in A_0'} N_t(a_0) \right) \).

Notice finally that for each \( b_0 \), the number of points in the intersection \( N_s(b_0) \cap \left( \bigcup_{a_0 \in A_0'} N_t(a_0) \right) \) (and therefore the number of Hausdorff elements at each location on the Hausdorff segment between \( A \) and \( B \)), depends only on the cardinality of the set \( A_0' \). This cardinality is not altered by varying the values of \( s \) and \( t \), implying that the Hausdorff segment from \( A \) to \( B \) has the same finite number of elements at every location between \( A \) and \( B \), and we have proved our theorem.

**Formal Proof of Theorem 4.1:**

**Proof.** Let \( A = \{a_1, a_2, \ldots, a_k\} \) and \( B = \{b_1, b_2, \ldots, b_m\} \) as above and \( n = k + m \). The proof is by induction on \( n \). We have already verified the cases \( n = 1, 2, \) and \( 3 \) directly. Assume \( n > 3 \). Let \( s \) be a number between 0 and \( r \), \( t = r - s \). The set \( (A + s) \cap (B + t) \) is an \( n - 1 \) point set \( X = \{x_1, x_2, x_3, \ldots, x_{n-1}\} \), with \( x_1 < x_2 < x_3 < \cdots < x_{n-1} \). Note that
\[
a_i < x_{2i-1} < b_i \quad \text{and} \quad b_i < x_{2i} < a_{i+1}.
\]

Each set \( C \) satisfying \( ACB \) with \( h(A, C) = s \) and \( h(B, C) = t \) will be a subset of \( X \). If \( x_1 \notin C \), then \( h(A, C) \geq d(a_1, x_2) > s \). So \( x_1 \notin C \). Similarly, we can show \( x_{n-1} \in C \). To find the remaining points in \( C \), we argue cases: \( x_2 \notin C \) and \( x_2 \in C \).

**Case I: \( x_2 \notin C \)**
Assume \( k > 1 \). Let \( A_1 = \{a_1\}, B_1 = \{b_1\} \) and let \( A^* = A - A_1, B^* = B - B_1 \). We will show that \( C^* = C - \{x_1, x_2\} \) satisfies \( h(A^*, C^*) = s \) and \( h(B^*, C^*) = t \). Note that since \( x_2 \notin C \), we must have \( x_2 \in C \) and \( x_3 \in C^* \). We will then have a one-to-one correspondence between the elements \( C \) on the segment joining \( A \) and the elements \( C^* \) on the segment joining \( A^* \) and \( B^* \). The inductive hypothesis tells us that there are \( F_{n-2} \) such elements \( C^* \) and, therefore, \( F_{n-2} \) elements \( C \).

First we show that \( C^* \) is not empty. If \( k = m \), then since \( h(B, C) = t \), it must be the case that \( C \) contains a point within \( t \) units of \( b_m \). Recall that \( C \subset X \), so the only such possible point is \( x_{n-1} \). If \( k = m + 1 \), then since \( h(A, C) = t \), it must be the case that \( C \) contains a point within \( t \) units of \( a_k \). Again, the only such possible point is \( x_{n-1} \). In either case, \( C \) must contain \( x_{n-1} \). Since \( x_{n-1} \notin C \), we must have \( x_{n-1} \in C^* \).

Now we show \( h(A^*, C^*) = s \). Let \( a_i \in A^* \) (\( i > 1 \)). Since \( h(A, C) = s \), there is a point \( c_i \in C \) so that
\[
d(a_i, c_i) \leq s.
\]
The only possible points in \( C \) satisfying (A.1) are \( x_{2i-2} \) and \( x_{2j-1} \). Since \( i > 1 \), it follows that \( 2i - 2 > 2 \), so neither \( x_{2i-2} \) nor \( x_{2j-1} \) is \( x_1 \) or \( x_2 \). Therefore, \( c_i \in C^* \) and \( d(a_i, C^*) = s \). Thus, \( d(A^*, C^*) = s \). A similar argument shows \( d(B^*, C^*) = t \). Now we show \( d(C^*, A^*) \leq s \). Let \( c \in C^* \). Then \( c \in C \). So there is a point \( a \in A \) so that \( d(c, a) \leq s \). Since \( C^* \subset X \), we must have \( c = x_i \) for some \( i > 2 \). Again, the only points in \( A \) that can be within \( s \) units of \( x_i \) are \( a_{i+1} \) if \( i \) is odd and \( a_{i+1} \) if \( i \) is even. Since \( i > 2 \), neither of these points can equal \( a_1 \). Therefore, \( a \in A^* \) and \( d(C^*, A^*) \leq s \). Therefore, \( h(C^*, A^*) = s \). Similarly, \( h(C^*, B^*) = t \).

**Case II: \( x_2 \in C \)**
In this case, we show that \( C^* = C - \{x_1\} \) satisfies \( A^*C^*B \) with \( h(A^*, C^*) = s \) and \( h(C^*, B) = t \). Again, this will give us a one-to-one correspondence between the elements \( C \) on the segment joining \( A \) and \( B \) and
the elements $C^*$ on the segment joining $A^*$ and $B$. There are exactly $F_{n-1}$ such sets $C^*$ by our inductive hypothesis, so there will be $F_{n-1}$ sets $C$.

The same argument as above shows that $C^*$ is not empty and that $d(A^*, C^*) = s$. To show $d(C^*, A^*) \leq s$, we argue as above. Let $c \in C^*$. Then $c \in C$. So there is a point $a \in A$ so that $d(c, a) \leq s$. Since $C \subseteq X$, we must have $c = x_i$ for some $i \geq 2$. Again, the only points in $A$ that can be within $s$ units of $x_i$ are $a_{i+1}$ if $i$ is odd and $a_{i+1}$ if $i$ is even. Since $i \geq 2$, neither of these points can equal $a_1$. Therefore, $a \in A^*$ and $d(C^*, A^*) \leq s$.

Now we show $h(C^*, B) = t$. Since $C \subseteq C^*$ and $d(C, B) = t$, we must have $d(C^*, B) \leq t$. Now we show $d(B, C^*) = t$. Let $b_j \in B$. If $j = 1$, we have $d(b_1, x_2) = t$, and $d(b_1, x_i) > t$ for all other $i > 2$. Therefore, $d(b_1, C^*) = t$. If $j > 1$, then there is a point $c \in C$ so that $d(b_j, c) \leq t$. Since $C \subseteq X$, we must have $c = x_i$ for some $i$. The only points in $C$ that can be within $t$ units of $b_j$ are $x_{2j-1}$ and $x_{2j}$. Since $j > 1$, neither of these points can equal $x_1$. Therefore, $c \in C^*$ and $d(B, C^*) \leq t$. Thus, $h(C^*, B) = t$.

Cases I and II show us that there are exactly $F_{n-2} + F_{n-1} = F_n$ elements at each location on the segment between $A$ and $B$.

**Formal verification of (4.1):**

**Proof.** Let $n \in \mathbb{N}$ and let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where $a_1$ is selected from the vertices of one $n$-gon and the point $a_i$ is the $i + 1$st vertex from $a_1$ moving clockwise on the same $n$-gon. On the second $n$-gon, which was rotated about the center, $b_1$ is the first vertex that lies $\frac{s}{n}$ degrees clockwise from $a_1$ and the point $b_i$ is the $j + 1$st vertex moving clockwise from $b_1$ on the same $n$-gon. Now let $d(a_i, B) = r = d(b_j, A)$ for each $i$ and let $0 \leq s \leq r$ and $t = r - s$. We will determine the number of points $C$ in $\mathcal{H}(\mathbb{R}^N)$ satisfying $ACB$ at each location between $A$ and $B$. Recall that $C$ is a subset of $(A + t) \cap (B + s)$.

Let $p(n)$ denote the number of points $C$ satisfying $ACB$ with $h(B, C) = s$. The proof of (4.1) is by induction on $n$. We begin with a few special cases:

- the "1-gon" (where $A$ and $B$ are each comprised of a single Euclidean point), and the "2-gon" (where $A$ and $B$ are the sets of endpoints of two line segments of equal length which are perpendicular bisectors of each other).

I. The simplest case occurs when $n = 1$, which was considered in [3]. In this case we have $p(1) = 1$.

II. Now suppose $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Note that $(A + t) \cap (B + s)$ is a four point set $X = \{x_1, x_2, x_3, x_4\}$, with $x_1$ between $a_1$ and $b_1$, $x_2$ between $b_1$ and $a_2$, $x_3$ between $a_2$ and $b_2$, and $x_4$ between $b_2$ and $a_1$. All elements $C$ that will satisfy $ACB$ will be subsets of $X$. If $x_1 \notin C$, then it must be that $x_3, x_4 \in C$. We notice the curve of alternating points, starting with $b_1$ and ending with $a_1$ while working clockwise, has the same restrictions for removing points from the set $C$ as the Fibonacci line of four points seen earlier. Thus we can conclude when $x_1 \notin C$ we will have $F_3 = 2$ elements $C$ which satisfy $ACB$. If $x_2 \notin C$, then a similar argument shows that there will be $2$ elements $C$ which satisfy $ACB$. Now, if $x_1, x_2 \notin C$, then we see that there will be another $3$ elements $C$ that satisfy $ACB$ and these elements will be $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$ and $\{x_1, x_2, x_3\}$. We can therefore conclude there are $7$ elements that satisfy $ACB$, thus $p(2) = 7$.

Now assume $n > 2$. Let $s$ be a number between $0$ and $r$ and let $t = r - s$. The set $(A + t) \cap (B + s)$ is an $2n$ point set $X = \{x_1, x_2, x_3, \ldots, x_{2n}\}$, where $x_{2i-1}$ is the point of intersection of the $t$-extension about $a_i$ and the $s$-extension about $b_i$ and $x_{2i}$ is the point of intersection of the $s$ extension about $b_i$ and the $t$ extension about $a_{i+1}$ for $i = 1, 2, \ldots, n$.

Each set $C$ satisfying $ACB$ with $h(A, C) = t$ and $h(B, C) = s$ will be a subset of $X$. To find the elements $C$, we argue cases: $x_1 \notin C$, $x_2 \notin C$ and $x_1, x_2 \in C$.

**Case I:** $x_1 \notin C$

In order to have $C$ satisfy $ACB$ we must have $d(a_1, C) = t$ and $d(b_1, C) = s$. This implies that $x_2, x_{2n} \in C$.

We now notice the curve of alternating points from $A$ and $B$, starting with $b_1$ and ending with $a_1$, is equivalent to a "Fibonacci line" of $2n$ points, which we have shown to have $F_{2n-1}$ elements satisfying $ACB$ by Theorem 4.1.

**Case II:** $x_2 \notin C$

This case can be argued in a similar manner as the previous case, thus we know that there will be an additional $F_{2n-1}$ elements which satisfy $ACB$. 

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Case III: $x_1, x_2 \in C$

We claim this case is similar to having a $2n + 1$ string of alternating points from $A$ and $B$, which by Theorem 4.1 will have $F_{2n}$ elements that satisfy $ACB$. By assumption we have $C = \{x_1, x_2\} \cup C'$, where $C'$ is a subset of $\{x_3, x_4, \ldots, x_{2n}\}$ such that if $x_i \not\in C'$ then $x_{i-1}$ or $x_{i+1} \in C'$ for $i = \{4, 5, \ldots, 2n - 1\}$. We can think of this as a string of alternating points starting with $b_1$, working in the clockwise direction, and ending with a new point $b_*$, where $b_* = b_1$, such that $x_1$ lies between $a_1$ and $b_*$. Then we see this is exactly the case when there is a line of $2n + 1$ alternating points as desired. Therefore, by Theorem 4.1, we have $F_{2n}$ elements which satisfy $ACB$.

Cases I, II and III show us that there are exactly $2F_{2n-1} + F_{2n}$ elements at each location on the segment between $A$ and $B$. The result $L_n = F_n + 2F_{n-1}$ follows easily from $L_n = F_{n+1} + F_{n-1}$ found in [6].