MULTIPLICITY RESULTS FOR SEMIPOSITONE PROBLEMS ON BALLS

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Dedicated to John Baxley on the occasion of his retirement from Wake Forest University.

ABSTRACT.

BRIEF ABSTRACT NEEDED

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1. INTRODUCTION

There is a vast literature devoted to semilinear Dirichlet problems of the form

\begin{equation}
\begin{cases}
-\Delta u = \lambda f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $f: \mathbb{R} \to \mathbb{R}$ is continuous, and $\lambda$ is a real parameter. A standard problem is to describe the dependence of solutions of (1.1) on the parameter $\lambda$, and one finds that (1.1) has solution continua \{$(\lambda, u)$\} of various shapes according to the properties of the nonlinearity $f$ and the underlying domain $\Omega$. A class of problems which is particularly relevant to the present paper is that of S–shaped bifurcation curves: when $f$ is sufficiently convex, for example, there is an interval $\Lambda := (\underline{\lambda}, \overline{\lambda})$ such that (1.1) has at least three solutions for any $\lambda \in \Lambda$ ([5]).

In this paper, we study a different structural condition on $f$ which guarantees the existence of at least three solutions of problem (1.1) when $\lambda \equiv 1$ and $\Omega$ is the unit
ball. Specifically, the nonlinearity $f$ in (1.1) will be modeled on the step function

\begin{equation}
\bar{f}(t) := \begin{cases} 
k, & \text{for } t \leq 1, \\
K, & \text{for } t > 1,
\end{cases}
\end{equation}

for constants $k < 0$ and $K > 0$. Theorem 4.1 states the conditions on $f$ precisely, but we emphasize that we do not impose any convexity or concavity conditions on $f$ (cf., e.g., [6],[7], where concavity assumptions are essential). The resulting problem is a semipositone ($f(0) < 0$) version of the problem considered in [8], in which $f$ is strictly positive. The latter paper, in turn, was motivated by some recent work by Henderson and Thompson ([10],[11],[12]) and Baxley et al. ([3],[4]), which we now describe.

In [12], Henderson and Thompson proved the following result.

**Theorem 1.1.** Let $0 < a < b < \frac{c}{2}$, and suppose that $f : [0, \infty) \to [0, \infty)$ is a continuous function satisfying

(i) $f(t) < 8a$, for $0 \leq t \leq a$,
(ii) $f(t) \geq 16b$, for $b \leq t \leq 2b$, and
(iii) $f(t) \leq 8c$, for $0 \leq t \leq c$.

Then the boundary-value problem

\begin{equation}
\begin{cases}
u'' + f(u) = 0, & \text{in } (0, 1), \\
\quad u(0) = u(1) = 0
\end{cases}
\end{equation}

has at least three distinct symmetric nonnegative solutions.

The constants 8 and 16 appearing in this theorem are, in fact, sharp, as can be seen by constructing explicit solutions of (1.3) (see, e.g., the introduction of [8]). In [3] and [4], Baxley et al. have shown that, with appropriate modifications of these bounds on $f$, one can obtain any desired odd number of positive solutions of (1.3). The subsequent paper [8] considers the PDE analogue of this problem, i.e., (1.1) for a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and $\lambda = 1$, and successfully generalizes Theorem 1.1 and its counterpart in [4] to higher dimensions.

As stated above, we are interested in (1.1) when the nonlinearity $f$ is semipositone, $f(0) < 0$. Arndt and Robinson have investigated this problem in one dimension and have found conditions guaranteeing the existence of three solutions, at least one of which is positive ([2]). Under slightly more restrictive conditions, they conclude that two of the three solutions are positive. In the hopes of gaining insight into this problem on more general domains and in higher dimensions, we consider problem (1.1) when $\lambda \equiv 1$ and $\Omega \subset \mathbb{R}^N$ is the unit ball, for $N \geq 2$. Since $\Omega$ is radially
symmetric, we concentrate on radial solutions of (1.1), thereby reducing the equation above to the ODE

\[ -u'' + \left( \frac{1 - N}{r} \right) u' = f(u), \]

where \( u = u(r) \) and \( r = |x| \). By the classical results of Gidas, Ni, and Nirenberg ([9]), any positive solution of (1.1) on the unit ball must be radial, but we will also consider nonpositive radial solutions of (1.1).

Since we are particularly interested in the possible relationship between \( k \) and \( K \), we carefully analyze the model problem (1.1) with the step nonlinearity \( \bar{f} \) above. Section 2 treats dimensions \( N \geq 3 \), and Section 3 then handles the problem in dimension \( N = 2 \). In either case, we will need to solve the elementary ordinary differential equation

\[ -u'' + \left( \frac{1 - N}{r} \right) u' = c, \]

for a given constant \( c \), and we find easily that the general solution of (1.5) is

\[ u(r) = \alpha + \beta r^{2-N} - \left( \frac{c}{2N} \right) r^2, \]

when \( N \geq 3 \), while

\[ u(r) = \alpha + \beta \ln r - \left( \frac{c}{4} \right) r^2 \]

when \( N = 2 \). The constants \( \alpha \) and \( \beta \) will be determined by the specifics of the boundary–value problems which arise.

Our methods in Sections 2 and 3 are entirely elementary and yield precise information on admissible pairs \( k \) and \( K \) for the model problems considered. As detailed in Section 4, these idealized problems produce lower solutions of (1.1), which we combine with other upper and lower solutions to obtain multiple solutions of (1.1).

2. THE MODEL PROBLEM ON A BALL, \( N \geq 3 \)

2.1. Explicit calculations. Let \( \bar{f} \) be the step function defined by (1.2). In this section, we will construct a bounded solution \( u \) of the model problem (1.1) for \( N \geq 3 \), where \( u \) has the form

\[
 u(r) = \begin{cases} 
 u_1(r), & \text{for } r_0 \leq r \leq 1, \\
 u_2(r), & \text{for } 0 \leq r \leq r_0, 
\end{cases}
\]

for \( u_1 \leq 1, u_2 \geq 1, \) and \( r_0 \in (0, 1) \). Since \( u \) is bounded, \( \bar{f}(u) \in L^2(0, 1) \) and elliptic regularity guarantees that \( u \in H^2(0, 1) \subset C^1(0, 1) \). Consequently, \( u_1 \) and \( u_2 \) must meet smoothly: \( u_1(r_0) = u_2(r_0) \) and \( u_1'(r_0) = u_2'(r_0) \).
Since \( u_1 \leq 1 \) on \((r_0, 1)\), \( \tilde{f}(u_1) = k \) and we have the boundary–value problem
\[
\begin{aligned}
-u_1'' + \left( \frac{1 - N}{r} \right) u_1' &= k, \\
\end{aligned}
\]  
(2.1)
\[
\begin{aligned}
 u_1(r_0) &= 1, \\
 u_1(1) &= 0.
\end{aligned}
\]

The general solution of the ODE in (2.1) is
\[
u_1(r) = \alpha_1 + \beta_1 r^{2-N} - \left( \frac{k}{2N} \right) r^2.
\]

Since \( u_1(1) = 0 \), we must have
\[
\alpha_1 + \beta_1 = \frac{k}{2N},
\]
so that \( u_1 \) has the form
\[
u_1(r) = \alpha_1 + \left( \frac{k}{2N} - \alpha_1 \right) r^{2-N} - \left( \frac{k}{2N} \right) r^2.
\]

As for \( u_2 \), we have the boundary–value problem
\[
\begin{aligned}
-u_2'' + \left( \frac{1 - N}{r} \right) u_2' &= K, \\
\end{aligned}
\]  
(2.2)
\[
\begin{aligned}
 u_2(r_0) &= 1.
\end{aligned}
\]

The solution of (2.2) has the form
\[
u_2(r) = \alpha_2 - \left( \frac{K}{2N} \right) r^2.
\]

Note that we omit the \( r^{2-N} \) term from \( u_2 \) since this term is singular at the origin.

Since
\[
u_1(r_0) = 1 = u_2(r_0),
\]

we have
\[
\alpha_1 + \left( \frac{k}{2N} - \alpha_1 \right) r_0^{2-N} - \left( \frac{k}{2N} \right) r_0^2 = 1 = \alpha_2 - \left( \frac{K}{2N} \right) r_0^2.
\]

Solving for \( \alpha_1 \) and \( \alpha_2 \) yields
\[
\alpha_1 = \frac{1}{1 - r_0^{2-N}} + \left( \frac{k}{2N} \right) \left( \frac{r_0^{2-N} - r_0^{2-N}}{1 - r_0^{2-N}} \right) < 0
\]
and
\[
\alpha_2 = 1 + \left( \frac{K}{2N} \right) r_0^2 > 0.
\]

We also have
\[
\beta_1 = \frac{1}{r_0^{2-N} - 1} + \left( \frac{k}{2N} \right) \left( \frac{1 - r_0^2}{1 - r_0^{2-N}} \right) > 0.
\]

We now impose the condition
\[
u_1'(r_0) = u_2'(r_0),
\]
which gives
\[ (2 - N) \beta_1 r_0^{1-N} - \left( \frac{k}{N} \right) r_0 = - \left( \frac{K}{N} \right) r_0. \]
Substituting the expression above for \( \beta_1 \) and simplifying, we obtain
\[
\left( \frac{k - K}{N(2 - N)} \right) r_0^N + \left( \frac{2K - Nk}{2N(2 - N)} \right) r_0^2 + 1 - \frac{k}{2N} = 0.
\]
Completing our construction of \( u \) thus rests on the existence of a root \( r_0 \in (0,1) \) of the polynomial
\[
P(r) := \left( \frac{k - K}{N(2 - N)} \right) r^N + \left( \frac{2K - Nk}{N(2 - N)} \right) r^2 + 1 - \frac{k}{2N}.
\]
Note that
\[ P(0) = 1 - \frac{k}{2N} > 1 = P(1), \]
so we know that \( P \) is positive at the endpoints \( r = 0 \) and \( r = 1 \). Moreover, since
\[ P'(r) = \left( \frac{k - K}{2 - N} \right) r^{N-1} + \left( \frac{2K - Nk}{N(2 - N)} \right) r, \]
we see that
\[ P'(0) = 0 \quad \text{and} \quad P'(1) = K - \frac{N}{N} > 0. \]
We need \( P \) to have a negative minimum inside the interval \((0,1)\). We see that \( P \) has two nonnegative critical points: \( r = 0 \) and the unique positive solution \( r_* \) of
\[
\left( \frac{k - K}{2 - N} \right) r_*^{N-2} + \left( \frac{2K - Nk}{N(2 - N)} \right) = 0, \quad \text{i.e.,}
\]
\[ r_*^{N-2} = \frac{2K - Nk}{N(K - k)} > 0. \]
Since \( N > 2, \ k < 0, \text{ and } K > 0 \), we find that
\[ \frac{2}{N} < r_*^{N-2} < 1. \]
Note also that
\[ P''(r_*) = \frac{2K - Nk}{N} > 0, \]
so that a minimum indeed occurs at \( r_* \).

Finally, we need \( P(r_*) \leq 0 \), which yields
\[
P(r_*) = \left( \frac{k - K}{N(2 - N)} \right) \left( \frac{2K - Nk}{N(K - k)} \right)^{N/(N-2)} + \left( \frac{2K - Nk}{2N(2 - N)} \right) \left( \frac{2K - Nk}{N(K - k)} \right)^{2/(N-2)} + 1 - \frac{k}{2N} \leq 0.
\]
Let \( r_* \) be any number such that (2.5) holds. We can then use (2.4) to relate \( k \) and \( K \),
\[
k = \left( \frac{N r_*^{N-2} - 2}{N r_*^{N-2} - N} \right) K,
\]
and combining this with (2.6) yields the estimates
\begin{equation}
K \geq \frac{2N(Nr_*^{N-2} - N)}{(2 - N)r_*^N + Nr_*^{N-2} - 2}
\end{equation}
and
\begin{equation}
k \leq \frac{2N(Nr_*^{N-2} - 2)}{(2 - N)r_*^N + Nr_*^{N-2} - 2}.
\end{equation}
Again, the bounds (2.8) and (2.9) hold for any chosen \( r_* \in (\frac{2}{N}, 1) \) and for any pair \((k, K)\) which satisfy (2.7) for this choice of \( r_* \).

Having determined an admissible pair \((k, K)\) corresponding to a chosen \( r_* \in (\frac{2}{N}, 1) \), we know that \( P(r) \) either has the isolated root \( r_* \) (if equality holds in both (2.8) and (2.9)) or two distinct roots \( r_1 \) and \( r_2 \) such that \( 0 < r_1 < r_* < r_2 < 1 \). In the first case, our construction above yields one solution \( u \) of (1.1) for the step function \( f \) defined by (1.2); in the second case, we obtain two such solutions. In either case, it remains to check the sign of the resulting solution(s). Since
\begin{align*}
\frac{u''}{(1 - N) \beta_1 r^N - \left( \frac{k}{N} \right)} > 0,
\end{align*}

it suffices to check the sign of \( u'_1(1) \): the solution \( u \) constructed from \( u_1 \) and \( u_2 \) will be positive on \([0, 1]\) if and only if \( u'_1(1) \leq 0 \).

Suppose that we choose \( k \) and \( K \) such that the polynomial \( P \) above has the unique root \( r_* \), for some \( r_* \in (\frac{2}{N}, 1) \). By a direct computation, we have
\begin{align*}
u'_1(1) &= (2 - N) \left[ \frac{1}{r_*^{2-N} - 1} + \left( \frac{k}{2N} \right) \left( \frac{1 - r_*^2}{1 - r_*^{2-N}} \right) \right] - \left( \frac{k}{N} \right),
\end{align*}
and \( u'_1(1) \leq 0 \) if and only if
\begin{align*}
(2 - N) \left[ 1 - \left( \frac{k}{2N} \right) (1 - r_*^2) \right] - \left( \frac{k}{N} \right) (r_*^{2-N} - 1) &\leq 0.
\end{align*}
Simplifying this inequality yields
\begin{align*}
\frac{2N(N - 2)r_*^{N-2}}{(2 - N)r_*^N + Nr_*^{N-2} - 2} &\leq k,
\end{align*}
which must be consistent with the precise value of \( k \) given by (2.9); thus, substituting this value of \( k \), we must verify that
\begin{align*}
\frac{2N(N - 2)r_*^{N-2}}{(2 - N)r_*^N + Nr_*^{N-2} - 2} &\leq \frac{2N(Nr_*^{N-2} - 2)}{(2 - N)r_*^N + Nr_*^{N-2} - 2}.
\end{align*}
Since both denominators are equal and negative, this last inequality holds because \( r_*^{N-2} \leq 1 \). In fact, \( r_*^{N-2} < 1 \), and we conclude from these calculations that the strict inequality \( u'_1(1) < 0 \) holds.

For an arbitrary admissible pair \((k, K)\), (2.4) determines the corresponding value of \( r_* \), and the distinct roots \( r_1 \) and \( r_2 \) of \( P(r) \) satisfy \( r_1 < r_* < r_2 \). Let \((k_*, K_*)\) denote the admissible pair for which \( P(r) \) has the isolated root \( r_* \), and let \( u_* \) denote
the corresponding solution constructed above. Let \( u_1 \) and \( u_2 \) denote the solutions corresponding to the meeting points \( r_1 \) and \( r_2 \), respectively (this will be confusing due to the preceding notation, in which \( u_1 \) and \( u_2 \) are the constituents of the solution \( u \); I should fix the notation). Since \( r_2 > r_* \), it follows easily that \( u_2 > u_* \) on \((0, 1)\), and we see that \( u_2 \) is positive. The sign of \( u_1 \), on the other hand, may change on the interval \((0, 1)\). Regardless, we find that there exists a positive solution of (1.1) for each admissible pair \((k, K)\).

2.2. Summary. For any dimension \( N \geq 3 \), define the curve \( \Gamma_N : \left(\frac{2}{N}, 1\right) \to \mathbb{R}^2 \) by

\[
\Gamma_N(t) := \left(\frac{2N(Nt^{N-2} - 2)}{(2 - N)t^N + Nt^{N-2} - 2}, \frac{2N(Nt^{N-2} - N)}{(2 - N)t^N + Nt^{N-2} - 2}\right).
\]

For each pair \((k, K)\) on or above the curve \( \Gamma_N \), there exists a positive solution of the model problem (1.1) on the unit ball with the step nonlinearity (1.2).

Figure 2.1 illustrates the dependence of these curves on the dimension \( N \), for dimensions 2 (covered in the next section), 3, 5, and 8.

3. THE MODEL PROBLEM ON A BALL, \( N = 2 \)

3.1. Explicit calculations. We now consider the case in which \( \Omega \) is the unit ball in \( \mathbb{R}^2 \), with \( \bar{f} \) defined by (1.2) as in the previous section. As before, we will construct a bounded solution \( u \) of (1.1) of the form

\[
u(r) = \begin{cases} u_1(r), & \text{for } r_0 \leq r \leq 1, \\ u_2(r), & \text{for } 0 \leq r \leq r_0, \end{cases}
\]

where \( u_1 \leq 1, u_2 \geq 1, \) and \( u_1 \) and \( u_2 \) meet smoothly at \( r_0 \in (0, 1) \).

Since \( u_1 \leq 1 \) on \((r_0, 1)\), \( \bar{f}(u_1) = k \) and we have the boundary–value problem

\[
\begin{cases} -u_1'' - \left(\frac{1}{r}\right) u_1' = k, \\ u_1(r_0) = 1, \quad u_1(1) = 0. \end{cases}
\]

(3.1)

The general solution of the ODE in (3.1) is

\[
u_1(r) = \alpha_1 + \beta_1 \ln r - \left(\frac{k}{4}\right) r^2.
\]

Since \( u_1(1) = 0 \), we must have

\[
\alpha_1 = \frac{k}{4},
\]

so that \( u_1 \) has the form

\[
u_1(r) = \frac{k}{4} + \beta_1 \ln r - \left(\frac{k}{4}\right) r^2.
\]
Figure 1. Admissible pair boundary curves $\Gamma_N$, for dimensions $N = 2, 3, 5, \text{ and } 8$. Admissible pairs $(k, K)$ lie on and above the corresponding curve $\Gamma_N$.

As for $u_2$, we have the boundary--value problem

\[
\begin{aligned}
& -u_2'' - \left( \frac{1}{r} \right) u_2' = K, \\
& u_2(r_0) = 1,
\end{aligned}
\]

whose solution has the form

\[
u_2(r) = \alpha_2 - \left( \frac{K}{4} \right) r^2.
\]

Note that we omit the logarithmic term from $u_2$ since this term is singular at the origin.

Since

\[u_1(r_0) = 1 = u_2(r_0),\]
we have
\[ \frac{k}{4} + \beta_1 \ln r_0 - \left( \frac{k}{4} \right) r_0^2 = 1 = \alpha_2 - \left( \frac{K}{4} \right) r_0^2. \]
Solving for \( \beta_1 \) and \( \alpha_2 \) yields
\[ \beta_1 = \frac{4 - k + kr_0^2}{4 \ln r_0} \]
and
\[ \alpha_2 = 1 + \left( \frac{K}{4} \right) r_0^2 > 0. \]
Imposing the condition
\[ u'_1(r_0) = u'_2(r_0), \]
we successively obtain
\[ \frac{\beta_1}{r_0} - \left( \frac{k}{2} \right) r_0 = - \left( \frac{K}{2} \right) r_0 \]
and
\[ \frac{4 - k + kr_0^2}{4r_0 \ln r_0} - \left( \frac{k}{2} \right) r_0 = - \left( \frac{K}{2} \right) r_0. \]
We must therefore find an \( r_0 \in (0,1) \) such that
\[ 4 - k + kr_0^2 = 2(k - K) r_0^2 \ln r_0, \]
i.e.,
the function
\[ P(r) := 4 - k + kr^2 - 2(k - K) r^2 \ln r \]
must vanish at some \( r_0 \in (0,1) \). Since
\[ P'(r) = 2kr - 2(k - K)(2r \ln r + r) = 4(K - k) r \ln r + 2Kr, \]
P has critical points at \( r = 0 \) and at the unique solution \( r_* \) of
\[ 4(K - k) \ln r = -2K, \]
so that
\[ r_* = \exp \frac{K}{2(k - K)}. \]
Since \( k - K < 0 \), we see that \( r_* \in (0,1) \). For any \( r_* \in (0,1) \) such that
\[ \ln r_* = \frac{K}{2(k - K)}, \]
we have
\[ (3.3) \quad k = \left( \frac{2 \ln r_* + 1}{2 \ln r_*} \right) K = \left( 1 + \frac{1}{2 \ln r_*} \right) K. \]
Since \( k < 0 \) and \( K > 0 \), we must have
\[ 1 + \frac{1}{2 \ln r_*} < 0, \]
from which we see that \( r_* \) must satisfy
\[ r_* > \frac{1}{\sqrt{e}}. \]
Calculating directly, we find that
\[ P(0) = 4 - k > 0, \quad P(1) = 4 > 0, \quad \text{and} \]
\[ P''(r_*) = 4(K - k)(\ln r_* + 1) + 2K = 8K - 4k > 0. \]
Thus, \( P \) attains a minimum at \( r_* \). In fact,
\[ P(r_*) = 4 - k + (k - K) \exp \left( \frac{K}{k - K} \right), \]
and, using the relationship (3.3), we have
\[ P(r_*) = 4 + \left( \frac{r_*^2 - 1}{2 \ln r_*} - 1 \right) K. \]
Since this minimum value must be nonpositive, the inequality
\[ 4 + \left( \frac{r_*^2 - 1}{2 \ln r_*} - 1 \right) K \leq 0 \]
must hold; simplifying yields the lower bound
\[ K \geq \frac{8 \ln r_*}{1 + 2 \ln r_* - r_*^2}, \quad \text{(3.4)} \]
from which we also obtain the upper bound
\[ k \leq \frac{8 \ln r_* + 4}{1 + 2 \ln r_* - r_*^2}. \quad \text{(3.5)} \]
Recall that the estimates (3.4) and (3.5) hold for any chosen \( r_* \in \left( \frac{1}{\sqrt{e}}, 1 \right) \).

3.2. Summary. The boundary curve of admissible pairs \((k, K)\) in dimension \(2\), \( \Gamma_2 : \left( \frac{1}{\sqrt{e}}, 1 \right) \to \mathbb{R}^2 \), is defined by
\[ \Gamma_2(t) := \left( \frac{8 \ln t + 4}{1 + 2 \ln t - t^2}, \frac{8 \ln t}{1 + 2 \ln t - t^2} \right). \]
As \( t \to \frac{1}{\sqrt{e}} \), points on the curve \( \Gamma_2 \) approach the point \((0, \frac{4}{e})\). For each pair \((k, K)\) on or above the curve \( \Gamma_2 \), there exists a positive solution of the model problem (1.1) on the unit ball with the step nonlinearity (1.2).

4. A THREE–SOLUTIONS THEOREM

Let us begin by recalling the definitions of upper and lower solutions. For a general domain \( \Omega \subset \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), a function \( u \in W^{2,p}(\Omega) \), for \( p > N \), is an upper solution of (1.1) if
\[ \Delta u + f(u) \leq 0, \quad \text{in} \quad \Omega, \]
\[ u \geq 0, \quad \text{on} \quad \partial \Omega. \quad \text{(4.1)} \]
One defines a lower solution of (1.1) by reversing both of these inequalities. As in the previous sections, we henceforth let \( \Omega \) denote the unit ball in \( \mathbb{R}^N \), for \( N \geq 2 \), and we focus on radial upper and lower solutions.
As above, $\Gamma_N$ denotes the boundary curve of admissible pairs $(k, K)$ for the model problem (1.1) with the step nonlinearity (1.2). For a fixed point $(k, K)$ on $\Gamma_N$, the construction in Sections 2 and 3 yields precisely one positive radial solution $\psi$ of the corresponding model problem, and the constant

$$M := \max_{x \in \Omega} \{ \psi(x) \}$$

satisfies $M > 1$. With this notation in place, we have our first multiplicity result.

**Theorem 4.1.** Let $(k, K)$ be a point on $\Gamma_N$, let $\psi$ be the positive radial solution of the corresponding model problem as above, and let $M$ be its maximum value. Let $a$, $b$, and $c$ be positive constants satisfying

$$0 < a < b < \frac{c}{K},$$

and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

(i) $ak < f(t) < 0$, for $t \leq a$,

(ii) $f(t) > bK$, for $b \leq t \leq Mb$, and

(iii) $ak < f(t) < c$ for all $t$.

Then the Dirichlet problem (1.1) has at least three radial solutions, two of which do not change sign in $\Omega$.

**Proof.** First, define the radial functions

$$\overline{u}_1(r) := 0 \quad \text{and} \quad \overline{u}_2(r) := \frac{c}{2N} \left( 1 - r^2 \right).$$

Since $f(0) < 0$, it is easy to see that $\overline{u}_1$ is an upper solution. On the other hand, since

$$-\Delta \overline{u}_2 = c \quad \text{in} \quad \Omega \quad \text{and} \quad \overline{u}_2 = 0 \quad \text{on} \quad \partial\Omega,$$

we have

$$\Delta \overline{u}_2 + f(\overline{u}_2) = -c + f(\overline{u}_2) < 0,$$

and we find that $\overline{u}_2$ is a strict upper solution.

The first lower solution can be obtained similarly. Define $\underline{u}_1$ by

$$\underline{u}_1(r) := \frac{ak}{2N} \left( 1 - r^2 \right),$$

so that

$$-\Delta \underline{u}_1 = ak \quad \text{in} \quad \Omega \quad \text{and} \quad \underline{u}_1 = 0 \quad \text{on} \quad \partial\Omega.$$

It follows that $\underline{u}_1$ is negative in $\Omega$ and, since

$$\Delta \underline{u}_1 + f(\underline{u}_1) = -ak + f(\underline{u}_1) > 0,$$

$\underline{u}_1$ is a strict lower solution.
Now define $u_2 := b\psi$. For points where $u_2 \leq b$, we have
\[ \Delta u_2 + f(u_2) = -kb + f(u_2) \geq k(a - b) \geq 0, \]
while, for points where $b \leq u_2 \leq bM$, we have
\[ \Delta u_2 + f(u_2) = -bK + f(u_2) \geq 0. \]
Consequently, $u_2$ is a lower solution. Moreover, we know that $u_2$ is positive in $\Omega$ since $\psi > 0$ in $\Omega$.

We therefore have two upper solutions and two lower solutions which clearly satisfy $\underline{u}_1 \leq \underline{v}_1 \leq u_1 \leq \underline{v}_2$. Applying standard degree–theoretic arguments (cf. [1],[13]) shows that (1.1) has three radial solutions $u_1$, $u_2$, and $u_3$ such that
\[ u_1 \leq u_1 \leq \underline{v}_1, \quad u_2 \leq u_2 \leq \underline{v}_2, \quad \text{and} \quad u_1 \leq u_3 \leq \underline{v}_2, \]
with $u_3(x_1) < \underline{u}_2(x_1)$ and $u_3(x_2) > \underline{v}_1(x_2)$ for some points $x_1$ and $x_2$. These three solutions are thus distinct. Moreover, we see that $u_1$ is negative and $u_2$ is positive, completing the proof.

$\square$

To obtain multiple positive solutions, we rely on the following observation, from which we directly obtain Corollary 4.3.

**Lemma 4.2.** Let $k < 0$ and $c > 0$ be fixed constants, and let $u$ be a radially symmetric solution of
\[ \Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega, \]
where $-k \leq f(u) \leq 0$ for $u \leq 1$ and $f(u) \leq c$ for all $u$. There is a $c^*$, depending only on $k$, such that if $c \leq c^*$, then $u$ is either strictly negative or strictly positive in $\Omega$, i.e. $u$ cannot be a sign-changing solution.

**Proof.** Consider the case where $u$ is sign-changing with $u'(1) > 0$, $u(r) < 0$ on some interval $(r_0, 1)$, $u$ is concave on $(r_1, 1]$ with $0 < r_1 < r_0 < 1$ with $u(r_1) = 1$, and $u > 1$ in $[0, r_1)$ with $u'(r) < 0$ in $(0, r_1)$ and $u'(0) = 0$.

For purposes of comparison, let $\bar{u}$ be a radially symmetric function satisfying $\Delta \bar{u} = -k$ in the unit ball minus the origin, with $\bar{u}(1) = 0$ and $\bar{u}'(1) = 0$. Let $r_k$ be such that $\bar{u}(r_k) = 1$, and note that $r_k \to 0$ as $k \to 0^-$. (We can solve for all of this explicitly as in the previous sections.)

A standard comparison shows that $u \leq \bar{u}$ in $[r_k, 1]$, so $0 < r_1 < r_k$. Moreover, by the concavity of $u$ we see that $|u'(r_1)| \geq \frac{1}{1-r_1}$, the absolute value of the slope of the line connecting $(r_1, 1)$ and $(1, 0)$. It follows via the Mean Value Theorem that
\[ u''(r) = \frac{-1}{r_1(1-r_1)} \] for some \( r \) in \((0, r_1)\). However, if \( c < \frac{1}{r_1(1-r_1)} \), then a contradiction is reached when we try to substitute into

\[ u'' + \frac{N-1}{r}u' = -f(u) \geq -c. \]

(Recall that \( u'(r) < 0 \).) Hence we can choose \( c^* \) as the minimum of \( \frac{1}{r(1-r)} \) for \( r \) in \((0, r_1)\). This minimum is just \( \frac{1}{r_1(1-r_1)} \) when \( r_1 \leq \frac{1}{2} \), and thus becomes very large as \( k \rightarrow 0^- \).

Corollary 4.3. In addition to the hypotheses of Theorem 4.1, suppose that \((k, K)\) is a point on \( \Gamma_N \) with \( k \) sufficiently close to zero to guarantee that \( c < c^* \). Then (1.1) has at least three radial solutions, two of which are positive and one of which is negative.

Proof. As in the proof of Theorem 4.1, we have a positive radial solution \( u_1 \), a negative radial solution \( u_2 \), and a third radial solution \( u_3 \). Since \( u_3(x_2) > u_1(x_2) = 0 \) at some \( x_2 \), Lemma 4.2 guarantees that \( u_3 \) is strictly positive. \( \square \)

In the case \( N = 1 \), Arndt and Robinson [2] have refined the result in Corollary 4.3 as follows: two of the three solutions are positive if either \( k > -8 \) or \(-24 < k < -8 \) and \( K < -\frac{16k}{k+8} \).

REFERENCES

