ON THE SECOND EIGENVALUE FOR NONHOMOGENEOUS QUASI-LINEAR OPERATORS

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Abstract. In this paper we study a nonlinear eigenvalue problem and associated perturbations of the problem. More specifically, we generalize a variational characterization of the second eigenvalue for homogenous quasi-linear elliptic operators, such as the $p$-Laplacian, to a class of non-homogeneous quasi-linear elliptic operators. Neumann boundary data is assumed throughout the paper. To demonstrate the utility of this characterization we use it to prove a generalized Fredholm alternative for nonresonant perturbations of the given eigenvalue problem.

Key words. second eigenvalue, quasi-linear, nonresonance

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1. Introduction. In this paper we consider the nonlinear eigenvalue problem

\[ Qu - \lambda |u|^{p-2}u = 0 \text{ a.e. in } \Omega, \]

\[ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega, \]

where $Q$ is a quasi-linear elliptic operator generalizing the $p$-Laplacian, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\nu$ is the unit outward normal on $\partial \Omega$, $\lambda$ is a real number, and $1 < p < \infty$. A key technical challenge is that $Q$ is not assumed to be homogeneous.

It will be clear that the principal eigenvalue is $\lambda_1 = 0$ with an associated simple eigenspace of constant functions, $W := \text{span}\{1\}$. Our interest is in establishing and exploiting an appropriate variational characterization for the second eigenvalue. For homogeneous operators, it is simplest to define $\lambda_2$ as the smallest number that is strictly larger than $\lambda_1$ such that (1.1) has a nontrivial solution, and it is well known that this definition has useful variational characterizations; see [2]. Choosing an appropriate definition becomes more subtle when dealing with nonhomogeneous operators, because the existence of a $\lambda$ such that (1.1) has a nontrivial solution does not necessarily imply the existence of an unbounded set of solutions. Describing unbounded sets of solutions is fundamental to understanding perturbations of (1.1). In order to motivate a useful definition, it is helpful to review the properties of more familiar operators.

In the linear case, e.g., where $Q = -\Delta$, it is well known that

\[ \lambda_2 = \inf_{v \in V_2 \setminus \{0\}} \frac{Q(v, v)}{||v||^2_{L^2}}, \]

where $Q(u, v) := \int_{\Omega} \nabla u \cdot \nabla v$ is the bilinear form associated with $Q$, and $V_2 := W^{-1} = \{ u \in W^{1,2}(\Omega) : \int_{\Omega} u = 0 \}$. An equivalent characterization, of minimax type, is given by

\[ \lambda_2 = \inf_{\gamma \in \mathcal{F}} \sup_{t \in [-1,1]} Q(\gamma(t), \gamma(t)), \]

where $\mathcal{F}$ is a suitable set of functions.
where $\Gamma := \{ \gamma : [-1, 1] \to \partial B_1 : \gamma \text{ is continuous, and } \gamma(\pm 1) = \pm \phi_1 \}$, $\partial B_1 := \{ u \in W^{1,2}(\Omega) : ||u||_{L^2} = 1 \}$, and $\phi_1 := (\frac{1}{|\Omega|})^{\frac{1}{2}}$.

These statements generalize in a natural way to certain homogeneous operators such as the $p$-Laplacian, i.e., $Qu := -\nabla \cdot (|\nabla u|^{p-2}\nabla u)$. To generalize (1.2) in this case, replace the bilinear form with the quasi-linear form $Q(u, v) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v$, and replace the subspace $V_2$ with the surface $V_p := \{ u \in W^{1,p}(\Omega) : \int |u|^{p-2} u = 0 \}$.

To this author’s knowledge, it was not known until recently whether replacing $V_2$ with $V_p$ is necessary. In fact, for the ODE case, and for the PDE case over certain simple domains such as $\Omega = [0,1]^N$, it is clear that one obtains the same value when minimizing over either $V_2$ or $V_p$. However, in [7] it is shown that, in general, a sharper characterization is obtained by minimizing over $V_p$.

To generalize (1.3), simply replace $L^2$ norms by $L^p$ norms in appropriate places. See [2], [3], [4], and the references therein for a more detailed discussion of the $p$-Laplacian and its eigenvalues.

For nonhomogeneous quasi-linear operators, there are many papers in the literature describing the properties of the principle eigenvalue and corresponding principle eigenfunctions. For example, see the results and references in [5]. However, there seem to be relatively few papers that consider the second eigenvalue. A notable exception is found in the work of Shapiro et al., where a variety of resonance and nonresonance theorems involving a second eigenvalue are proved under very general circumstances. For example, see [10] and [12] and references therein. (Studying these results was a primary motivation for this paper.) In both [10] and [12], the second eigenvalue is defined as

$$\lambda^*_2 := \lim \inf_{||v||_{L^p(\Omega)} \to \infty} \frac{Q(v,v)}{||v||_{L^p(\Omega)}^p} \text{ for } v \in V_2.$$  

This definition is naturally motivated by the nearly orthogonal splitting of the Banach space $W^{1,p}(\Omega) = W + V_2$. Observe that this definition generalizes (1.2) but does not substitute $V_p$ for $V_2$. The result in [7] shows that $\lambda^*_2 < \lambda_2$ in general. It follows that our results, which are based upon the sharper characterization, lay the groundwork for more general existence theorems, as is demonstrated in section 4.

The paper is organized as follows. In section 2 we provide a precise description of $Q$ along with preliminary remarks on notation and simple properties. In section 3 we define $\lambda_2$ using a natural generalization of (1.3) and then show that this is equivalent to a natural generalization of (1.2). Moreover, we establish a helpful estimate regarding the primitive of the quasi-linear form associated with $Q$ and hint at possible generalizations. Finally, in section 4 we consider the boundary value problem

$$Qu - g(x,u) = h \text{ a.e. in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega$$

and prove an existence theorem assuming that $\frac{g(x,u)}{|u|^{p-2}u}$ lies strictly between 0 and $\lambda_2$ for large $u$ and that $h \in (W^{1,p}(\Omega))^*$ is arbitrary. This demonstrates the usefulness of $\lambda_2$ as a bound for existence theorems and generalizes one case of the Fredholm alternative for self-adjoint linear operators. The proof obtains a solution as a saddle point over linked sets. For the relevant definitions and theorems of measure theory, Sobolev spaces, and variational theory, we refer the reader to [9], [1], and [13], respectively.
2. Preliminaries. Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that

(A-1) (Carathéodory) The map $x \to A(x, \xi)$ is measurable for each $\xi \in \mathbb{R}^N$, and the map $\xi \to A(x, \xi)$ is continuous for a.e. $x \in \Omega$.

(A-2) (Growth) There exist a positive constant $c_1$, a constant $p \in (1, \infty)$, and a nonnegative function $\tilde{h} \in L^{p'}(\Omega)$, where $p' = p/(p - 1)$, such that

$$|A(x, \xi)| \leq \tilde{h}(x) + c_1|\xi|^{p-1}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

(A-3) (Ellipticity) There exists a positive constant $c_2$ such that

$$\sum_{i=1}^{N} A_i(x, \xi) \cdot \xi \geq c_2 \sum_{i=1}^{N} |\xi|^p$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p$ is as in (A-2).

(A-4) (Monotonicity) Assume that for a.e. $x \in \Omega$ and each $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$,

$$\sum_{i=1}^{N} [A_i(x, \xi) - A_i(x, \xi^*)](\xi_i - \xi_i^*) > 0.$$

(A-5) (One-sided $p$-homogeneity) $A(x, t\xi) \cdot \xi \leq t^{p-1}A(x, \xi) \cdot \xi$ for all $t > 0$ and all $(x, \xi) \in \Omega \times \mathbb{R}^N$.

Given the assumptions above, it now makes sense to formally define

$$Qu := -\nabla \cdot (A(x, \nabla u))$$

and to define the quasi-linear Dirichlet form

$$Q(u, v) := \int_{\Omega} A(x, \nabla u) \cdot \nabla v \quad \forall u, v \in W^{1,p}(\Omega).$$

In view of (A-2), we see that $Q$ is well defined on $W^{1,p}(\Omega) \times W^{1,p}(\Omega)$.

In order to impose a variational structure on $Q$, we assume that $A(x, \xi) = \nabla \xi F(x, \xi)$, where $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following:

(F-1) (Carathéodory) The map $x \to F(x, \xi)$ is measurable for each $\xi \in \mathbb{R}^N$, and the map $\xi \to F(x, \xi)$ is continuously differentiable for a.e. $x \in \Omega$.

(F-2) (Growth) There exist a positive constant $c_3$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$|F(x, \xi)| \leq h(x) + c_3|\xi|^p$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, where $p$ is chosen as in (A-2).

(F-3) (Normalization) $F(x, 0) = 0$ for a.e. $x \in \Omega$.

It follows that $u \mapsto \int_{\Omega} F(x, \nabla u)$ is a $C^1$ functional on $W^{1,p}(\Omega)$ with derivative $u \mapsto Q(u, \cdot)$. Moreover, using the fundamental theorem of calculus and Fubini's theorem, we see that $\int_{\Omega} F(x, \nabla u) = \int_0^1 Q(tu, u)dt$, which will be the more useful form for later estimates. Given this structure, we see that solutions of (1.1) and (1.4) are equivalent to critical points of the functionals

$$E_\lambda(u) = \int_0^1 Q(tu, u)dt - \frac{\lambda}{p} \int_{\Omega} |u|^p$$

and

$$J(u) = \int_0^1 Q(tu, u)dt - \int_{\Omega} G(x, u) - h(u),$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, where $p$ is chosen as in (A-2).
respectively, where \( G(x, u) := \int_0^u g(x, t) dt \).

Throughout this paper we will use the norm in \( W^{1,p}(\Omega) \) given by

\[
\|u\|_{1,p}^p = \|u\|_{L^p}^p + \sum_{i=1}^N \| \frac{\partial u}{\partial x_i} \|_{L^p}^p,
\]

where \( \| \cdot \|_{L^p} \) denotes the \( L^p(\Omega) \) norm. We will also be using the seminorm

\[
|u|_{1,p} = \left\{ \sum_{i=1}^N \| \frac{\partial u}{\partial x_i} \|_{L^p} \right\}^{1/p}.
\]

Observe that by the definition of \( Q \) in (2.2) and (A–3) we get

\[
Q(u, u) \geq c_2 \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^p = c_2 (|u|_{1,p}^p)
\]

for all \( u \in W^{1,p}(\Omega) \) so that \( \lim \inf_{\|u\|_{L^p} \to \infty} \frac{Q(u, u)}{\|u\|_{L^p}^p} \geq 0 \). Define

\[
\lambda_1 := \lim \inf_{\|u\|_{L^p} \to \infty} \frac{Q(u, u)}{\|u\|_{L^p}^p}
\]

as in [11, p. 1821]. Since \( Q(u, u) = 0 \) for \( u \) constant, we see that \( \lambda_1 = 0 \). On the other hand, for nonconstant \( v \in W^{1,p}(\Omega) \) we obtain from (2.4) that \( Q(v, v) > 0 \), so \( \lambda_1 = 0 \) behaves like a simple eigenvalue with constant normalized eigenfunction \( \phi_1 = \frac{1}{|\Omega|^{1/p}} \) and corresponding eigenspace \( W := \text{span}\{1\} \).

Remark 1. Conditions (A–1) through (A–5) and (F–1) through (F–3) are not entirely independent. For example, the growth condition on \( F \) can be derived from the growth condition on \( A \). For convenience and clarity, we chose to state these standard properties separately.

Remark 2. Condition (A–5) is only used in the proof of the Palais–Smale condition in section 4. Conditions of this type have appeared often in the literature. Recently, in [12], Shapiro used a similar condition to prove an existence theorem for (1.4), assuming that \( g \) satisfies a superlinear growth condition. (A–5) implies that \( pF(x, \xi) \leq \nabla_\xi F(x, \xi) \cdot \xi \), which is closely related to the Ambrosetti–Rabinowitz condition (AR), which has appeared in a variety of contexts. However, AR is usually imposed on the nonlinear perturbation of a linear elliptic boundary value problem (see condition (p_4) on page 9 of [6]), whereas (A–5) is imposed upon the nonlinear differential operator.

3. Definition and characterization of \( \lambda_2 \). We begin by stating a natural generalization of (1.3).

**Definition 3.1.** Let \( \Gamma_R := \{ \gamma : [-1, 1] \to \partial B_R : \gamma \text{ is continuous, and } \gamma(\pm 1) = \pm \phi_R \} \), where \( \partial B_R := \{ u \in W^{1,p}(\Omega) : \|u\|_{L^p} = R \} \), and \( \phi_R := (\frac{R}{|\Omega|})^{\frac{1}{p}} \). Define

\[
\lambda_{2,R} := \inf_{\gamma \in \Gamma_R} \sup_{t \in [-1, 1]} \frac{Q(u, v)}{|v|_{L^p}^p},
\]

and \( \lambda_2 := \lim \inf_{R \to \infty} \lambda_{2,R} \).

Our first lemma establishes an equivalence between the above definition and a generalization of (1.2).

**Lemma 3.2.** \( \lambda_2 = \lim \inf_{\|v\|_{L^p} \to \infty} \frac{Q(v, u)}{|v|_{L^p}^p} \) for \( v \in V_p := \{ u \in W^{1,p}(\Omega) : \int_{\Omega} |u|^{p-2}u = 0 \} \).
Proof. Consider definition (3.1). Any curve in $\Gamma_R$ must cross $V_p$, so it is clear that

$$\lambda_{2,R} \geq \inf_{v \in V_p \cap \partial B} \frac{Q(v,v)}{R^p}.$$  

Thus $\lambda_2 \geq \lim \inf_{\|v\|_{L^p} \to \infty} \frac{Q(v,v)}{\|v\|_{L^p}^p}$ for $v \in V_p$.

On the other hand, given $\epsilon > 0$ and $R > 0$, consider $v_R \in V_p \cap \partial B$ such that

$$\frac{Q(v,v)}{R^p} < \inf_{v \in \partial B \cap V_p} \frac{Q(v,v)}{R^p} + \epsilon.$$  

Now consider the curve

$$\eta(t) := \begin{cases} 
-T + tv_R \text{ for } 0 \leq t \leq 1, \\ 
t - T + v_R \text{ for } 1 \leq t \leq 2T + 1, \\ 
T + (-t + 2T + 2)v_R \text{ for } 2T + 1 \leq t \leq 2T + 2,
\end{cases}$$

where $T$ is a large positive constant. This is essentially a long line segment paralleling $W$ with short connections to $W$ on either end. Also consider the set

$$C := \{ u \in W^{1,p}(\Omega) : \exists \text{ continuous } \gamma : [-1,1] \to \partial B_{\|u\|_{L^p}} \text{ such that }$$

$$\gamma(-1) = -\frac{\|u\|_{L^p}}{\|\gamma\|_{L^p}} \text{ and } Q(\gamma(t),\gamma(t)) < \lambda_{2,\|u\|_{L^p}}\|u\|_{L^p}^p \forall t \}. $$

$C$ is the set of points in $W^{1,p}(\Omega)$ that can be connected to a negative constant function by a curve which stays on the surface of an $L^p$ sphere, $\partial B_R$, without crossing a point where $\frac{Q(u,v)}{\|v\|_{L^p}^p} \geq \lambda_{2,R}$. A straightforward argument shows that $C$ is open. It is clear that $\eta(0) = -T \in C$. Thus $\eta^{-1}(C)$ is a nonempty open subset of $[0,2T + 2]$. If $\eta^{-1}(C) = [0,2T + 2]$, then $\eta(2T + 2) = T$ is in $C$, so $T$ can be connected to $-T$ by a curve on $\partial B_R$, where $\frac{Q(u,v)}{\|v\|_{L^p}^p} < \lambda_{2,R}R^p$, for $R = \frac{T}{\|\gamma\|_{L^p}}$, which contradicts the definition of $\lambda_{2,R}$.

Therefore, there is a maximal $t' \in (0,2T + 2)$ such that $\eta(t) \notin C$ for $t \in [0,t')$. Let $u' = \eta(t')$ and $R' = \|u'\|_{L^p}$. We see that $Q(u',u') \geq \lambda_{2,R'}(R')^p$, or else we could move a little farther along $\eta$ while remaining in $C$, which would contradict our choice of $t'$. For large $T$, we argue that $t' \in (1,2T + 1)$. For any $t \in [0,1] \cup [2T + 1,2T + 2]$, we have $Q(\eta(t),\eta(t)) = \frac{Q(t^*v_R,t^*v_R)}{\|\eta(t)\|_{L^p}^p}$, where $t^* \in [0,1]$ and $\|\eta(t)\|_{L^p} \to \infty$ as $T \to \infty$, so

$$\frac{Q(\eta(t),\eta(t))}{\|\eta(t)\|_{L^p}^p} < \lambda_2 \text{ for large } T.$$  

This estimate leads to the fact that $\eta(t) \in C$ for any $t \in [0,1]$ and $\eta(t) \notin C$ for any $t \in [2T + 1,2T + 2]$, and thus $t' \in (1,2T + 1)$. Along the segment where $1 < t < 2T + 1$, we see that $Q(\eta(t),\eta(t)) = Q(v_R,v_R)$. Also, since for any $u \in W^{1,p}(\Omega)$ the function $\tau \to \int_0^\tau |u + \tau|^p$ achieves a unique minimum for $s$ such that $u + s \in V_p$, we see that $\|\eta(t)\|_{L^p}^p$ achieves its minimum at $t = T + 1$, where $\eta(t) = v_R$. Hence $R' > R$ and $\lambda_{2,R'} \leq \frac{Q(u',u')}{(R')^p} \leq \frac{Q(v_R,v_R)}{R^p} \leq \inf_{V_p \cap \partial B_R} \frac{Q(v,v)}{R^p} + \epsilon$.

The lemma follows. \[\square\]

Of course, if the given characterizations lead to $\lambda_2 = \lambda_1 = 0$, then the existence theorem in section 4 would not be very interesting. Thus we should take a moment to mention the following.

Lemma 3.3. $0 < \lambda_2 < \infty$.

Proof. The first inequality follows from the ellipticity condition and a Poincare–type inequality. For the details of a more general estimate see Lemma 3.2 in [8]. The second inequality follows from the growth condition on $A$. \[\square\]

In the homogeneous case the relationship between $Q$ and its primitive is trivial, because $\int_0^t Q(tu,u)dt = Q(u,u) \int_0^t t^{p-1}dt = \frac{1}{p}Q(u,u)$, but in the nonhomogeneous
case this relationship is more subtle. The following lemma provides a useful estimate and a relationship between $Q$ and its primitive.

**Lemma 3.4.** $\lambda_2 \leq \lim \inf_{\|v\|_{L^p} \to \infty} \frac{p \int_0^1 Q(tv,v)dt}{\|v\|^p}$ for $v \in V_p$.

**Proof.** Given $\epsilon > 0$ and $\delta > 0$, choose $R > 0$ such that $\inf_{\partial B_r \cap V_p} Q(v,v) \geq (\lambda_2 - \epsilon)r^p$ for all $r > \delta R$. Hence for $v \in (\partial B_R \cup V_p)$,

$$p \int_0^1 Q(tv,v)dt \geq p(\lambda_2 - \epsilon) \int_\delta^1 t^{p-1} dt \int_\Omega |v|^p = (\lambda_2 - \epsilon)(1 - \delta^p) \int_\Omega |v|^p.$$

The result follows. \qed

**Remark 3.** An interesting possibility would be to base our definition of $\lambda_2$ on the generalized Raleigh quotient

$$\frac{p \int_0^1 Q(tv,v)dt}{\int_\Omega |v|^p}$$

rather than on

$$\frac{Q(u,u)}{\int_\Omega |u|^p}.$$ 

This is not an issue in the homogenous case, but might be of interest for certain nonhomogeneous operators. Notice that a straightforward Lagrange multipliers argument shows that a critical point of $\int_0^1 Q(tv,v)dt$ constrained to a sphere $\partial B_R$ must occur on $V_p \cap \partial B_R$. Hence a minimax characterization similar to Definition 3.1 would reduce to the lim inf characterization in Lemma 3.4. Moreover, it would be of interest to compare these variational definitions to an arguably more natural definition generalizing $\lambda_2 := \inf\{\lambda > 0 : (1.1)\}$ has a nontrivial solution}, as in [2].

**Remark 4.** In order to simplify notation and clarify exposition, we have limited ourselves to second order operators. However, the notation and the results in this paper generalize in a straightforward way to operators of order $2m$. See [10] for details.

**4. A nonresonance theorem.** In this section we consider the boundary value problem (1.4), where $\frac{g(x,u)}{|u|^{p-2}u}$ is bounded strictly between the eigenvalues $\lambda_1$ and $\lambda_2$, and where $h \in (W^{1,p}(\Omega))^*$. This is called a nonresonance problem and we should expect, as in the Fredholm alternative, that the problem will be solvable for any choice of $h$. Proving this theorem verifies the practicality of Definition 3.1.

First we set the stage for a variational proof. Let $G(x,u) := \int_0^u g(x,t)dt$ and let

$$J(u) := \int_0^1 Q(tu,u)dt - \int_\Omega G(x,u) - h(u) \text{ for } u \in W^{1,p}(\Omega).$$

$J$ is a $C^1$ functional with

$$J'(u)v = Q(u,v) - \int_\Omega g(x,u)v - h(v).$$

Critical points of $J$ correspond to weak solutions of (1.4).

Our proof establishes the existence of a critical point using a saddle point theorem over linked sets. See [13, Theorem 8.4] for details. We will show that $J$ has a saddle
geometry over the linked sets \( W \) and \( V_p \), and then we will show that \( J \) satisfies the Palais–Smale condition, i.e., that if \( \{u_n\} \subset W^{1,p}(\Omega) \) such that \( \{J(u_n)\} \) is bounded and \( J'(u_n) \to 0 \) in \( (W^{1,p}(\Omega))^* \), then \( \{u_n\} \) has a converging subsequence.

**Theorem 4.1.** Assume (A–1), (A–2), (A–3), (A–4), (A–5), (F–1), (F–2), and (F–3) and that \( Q(u) := -\nabla \cdot (A(x,\nabla u)) \). In addition assume that \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function satisfying

\[
0 \leq \epsilon \leq \frac{g(x,u)}{|u|^{p-2}u} \leq \lambda_2 - \epsilon
\]

for some \( \epsilon > 0 \). Assume that \( h \in (W^{1,p}(\Omega))^* \). Then (1.4) has at least one weak solution in \( W^{1,p}(\Omega) \).

**Proof.** It is clear that, for any constant \( C > 0 \), if \( \gamma : [-1,1] \to W^{1,p}(\Omega) \) is a continuous curve such that \( \gamma(\pm 1) = \pm C \), then there is at least one \( t_0 \in [-1,1] \) such that \( \gamma(t) \in V_p \). Thus \( \{\pm C\} \) and \( V_p \) link.

Consider \( J \) restricted to \( W \). Since \( Q(u,c) = 0 \) for any \( u \in W^{1,p}(\Omega) \) and any constant \( c \), we have

\[ J(c) = - \int_\Omega G(x,c) - h(c). \]

Using (1.2) we see that \( G(x,c) \geq \frac{c}{p} |c|^p \) for all \( c \), so it follows that \( \lim_{||w||_{W^{1,p}(\Omega)} \to \infty} J(w) = -\infty \).

Consider \( J \) restricted to \( V_p \). First we choose \( \delta > 0 \) and apply ellipticity to get

\[ J(v) \geq \delta c_2 \int_\Omega |\nabla v|^p + (1 - \delta) \int_0^1 Q(tv,v)dt - \int_\Omega G(x,v) - h(v). \]

By Lemma 3.4 there is an \( R > 0 \) such that \( \int_0^1 Q(tv,v) \geq (\lambda_2 - \frac{\epsilon}{p}) ||v||_{L^p}^p \) for \( ||v||_{L^p} > R \). Applying (1.4) again, we see that \( G(x,v) \leq \frac{\lambda_2 - \epsilon}{p} |v|^p \) for all \( v \), so

\[ J(v) \geq \delta c_2 \int_\Omega |\nabla v|^p + (1 - \delta) \left( \frac{\lambda_2 - \frac{\epsilon}{p}}{p} \right) ||v||_{L^p}^p \geq \lambda_2 - \frac{\epsilon}{p} ||v||_{L^p}^p - \|h\|_{W^{1,p}(\Omega)}^p \|

For \( \delta \) small enough, there is a constant \( c' > 0 \) such that

\[ J(v) \geq c' ||v||_{1,p}^p - ||h||_{W^{1,p}(\Omega)}^p ||v||_{1,p}. \]

Hence, for \( J \) restricted to \( V_p \), we have \( \lim_{||v||_{1,p} \to \infty} J(v) = \infty \).

We have shown that \( J \) has a saddle geometry over the linked sets \( W \) and \( V_p \). It remains to prove the Palais–Smale condition. Suppose that \( \{u_n\} \subset W^{1,p}(\Omega) \) such that \( ||J(u_n)|| \leq K \) for all \( n \) for some \( K > 0 \), and such that \( J'(u_n) \to 0 \) in \( (W^{1,p}(\Omega))^* \). We must show that \( \{u_n\} \) has a converging subsequence. We note that it suffices to show that there is a bounded subsequence. See [10]. Suppose that \( ||u_n||_{1,p} \to \infty \).

First, using ellipticity and (1.4), we see that

\[ J'(u_n) \cdot u_n \geq c_2 (||u_n||_{1,p}^p - (\lambda_2 - \epsilon)||u_n||_{L^p}^p. \]

Since \( J'(u_n) \to 0 \), we can divide the given inequality by \( ||u_n||_{L^p} \) and discover that \( ||u_n||_{1,p} \leq c' ||u_n||_{L^p} \) for some \( c' > 0 \), and thus \( ||u_n||_{1,p} \leq c'' ||u_n||_{L^p} \) for some \( c'' > 0 \).
Without loss of generality, the sequence \( \{ \frac{u_n}{||u_n||_{L^p}} \} \) converges weakly in \( W^{1,p}(\Omega) \) and strongly in \( L^p(\Omega) \) to some function \( u \) such that \( ||u||_{L^p} = 1. \) Now consider

\[
J'(u_n) \cdot 1 = \int_\Omega g(x, u_n) = \int_\Omega g(x, u_n^+) + \int_\Omega g(x, -u_n^-).
\]

Equation (4.1) implies

\[
\epsilon \frac{\int_\Omega |u_n^+|^{p-1}}{||u_n||_{L^p}^{p-1}} \leq \frac{\int_\Omega g(x, \pm u_n^+)}{||u_n||_{L^p}^{p-1}} \leq (\lambda_2 - \epsilon) \frac{\int_\Omega |u_n^+|^{p-1}}{||u_n||_{L^p}^{p-1}},
\]

and \( J'(u_n) \to 0 \) implies that

\[
\lim_{n \to \infty} \frac{\int_\Omega g(x, u_n^+)}{||u_n||_{L^p}^{p-1}} = -\lim_{n \to \infty} \frac{\int_\Omega g(x, -u_n^-)}{||u_n||_{L^p}^{p-1}},
\]

where we can assume that the given limits exist by passing to a subsequence. If this limit is 0, then \( \int_\Omega |u^+|^{p-1} = \int_\Omega |u^-|^{p-1} = 0, \) which contradicts the fact that \( u \) is nontrivial. Thus the limit is positive and it follows that both \( u^+ \) and \( u^- \) are nontrivial. Now consider

\[
J'(u_n) \cdot u_n^+ = Q(u_n, u_n^+) - \int_\Omega g(x, u_n)u_n^+ = Q(u_n^+, u_n^+) - \int_\Omega g(x, u_n^+)u_n^+.
\]

Dividing through by \( ||u_n||_{L^p}^p \), using (4.1), and using the fact that \( J'(u_n) \cdot \frac{u_n}{||u_n||_{L^p}} \to 0, \) we see that

\[
Q(u_n^+, u_n^+) \leq \left( \lambda - \frac{\epsilon}{2} \right) \int_\Omega |u_n^+|^p
\]

for \( n \) large. A similar estimate holds for \( u_n^- \). Hence, for large \( n \), we have that \( u_n^+ \) and \( u_n^- \) are nontrivial and satisfy the previous inequality. Using such a \( u_n \), we construct the curve \( \gamma(\alpha, \beta) = \alpha u_n^+ - \beta u_n^- \) such that \( \alpha \) and \( \beta \) are nonnegative and \( \alpha^p||u_n^+||_{L^p}^p + \beta^p||u_n^-||_{L^p}^p = ||u_n||_{L^p}^p. \) It is now straightforward to check that this curve lives on the \( L^p \) ball of radius \( ||u_n||_{L^p} \) and crosses \( V_p. \) Using (A–5) we get the estimate

\[
Q(\alpha u_n^+ - \beta u_n^-, \alpha u_n^+ - \beta u_n^-) = Q(\alpha u_n^+ - \beta u_n^-) + Q(-\beta u_n^- \beta u_n^-) \leq (\alpha^p||u_n^+||_{L^p}^p + \beta^p||u_n^-||_{L^p}^p) (\lambda_2 - \frac{\epsilon}{2}) \leq ||u||_{L^p}(\lambda_2 - \frac{\epsilon}{2}).
\]

However, for large \( ||u_n||_{L^p} \), we must have \( Q(\gamma(\alpha, \beta)) \geq (\lambda_2 - \frac{\epsilon}{2})||u_n||_{L^p}^p \) at the point where this curve crosses \( V_p. \) Thus we have reached a contradiction and the proof is complete. \( \square \)

REFERENCES


