APPLICATIONS OF RECURRENCE BOUNDS
TO NETWORKS AND PATHS

Kenneth S. Berenhaut\textsuperscript{1,3}, John D. Foley\textsuperscript{2}
\textsuperscript{1,2}Department of Mathematics
Wake Forest University
P.O. Box 7388, Winston-Salem, NC 27109, USA
\textsuperscript{1}e-mail: berenhks@wfumc.edu
\textsuperscript{2}e-mail: folejd4@wfumc.edu

Abstract: This note provides an equivalence between bounds for linear recurrences and results for a model of network flow, wherein a signal is propagated over a series of repeaters with varying configurations. Recent recurrence results are then reinterpreted within this framework. Extensions to higher dimensions are also considered.

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1. Introduction

We consider here a model for directed signal transmission via a series of repeaters. For our purposes, repeaters are simply devices which are capable of receiving a signal on one frequency and transmitting it on another. We will demonstrate an equivalence between bounds for linear recurrences and optimization results for such systems. The many results proven for linear recurrences in the past few years can then be brought to bear in this situation.

Before delving into particulars we provide the following analogy, which is also covered by the theory developed here.
Example 1. Suppose one is laying out equally sized concrete slabs end-to-end in a line from one side of a room to the other. In addition, suppose that each slab is suitable for either a left foot or a right foot (but never both). Given that a person's stride is only able to cover 5 slabs, what is the configuration of slabs that maximizes the number of possible paths across the room, where no back-steps are allowed?

Returning to our situation, for $k \geq 1$, consider linear recurrences of the form

$$b_n = \sum_{i=1}^{k} \alpha_{n,n-i} b_{n-i},$$

where $b_i = 0$ for $i \leq 0$, and

$$\alpha_{n,i} \in [-p,0],$$

for $n-k \leq i \leq n$ and $n \geq 2$. Without loss of generality we will assume that $b_1 \in \{-1,1\}$.

Now, define

$$U_n = U_n(k,p) \overset{df}{=} \max\{|b_n| : \{b_i\}, \{\alpha_{n,i}\} \text{ satisfy (1) and (2)}\}; \ n \geq 2. \ (3)$$

Much is now known regarding behavior of $\{U_i(k,p)\}$ and related sequences (cf. [1]-[6], and the references therein). The results often imply applicable bounds for matrix inverses and reciprocals and zeros of power series.

Recurrences with varying or random coefficients have been studied by many previous authors. A partial survey of such literature contains Viswanath [13] and [14], Viswanath and Trefethen [15], Embree and Trefethen [7], Wright and Trefethen [16], Mallik [10], Popenda [12], Kittapa [9], and Odlyzko [11].

One particularly interesting result for $\{U_i(2,p)\}$ is the following.

Theorem 1. The following hold:

(i) If $p < (1/3)^{1/3}$, then $\{U_i(2,p)\}$ tends to zero at an exponential rate.

(ii) If $p > (1/3)^{1/3}$, then $\{U_i(2,p)\}$ tends to infinity at an exponential rate.

(iii) If $p = (1/3)^{1/3}$, then $\{U_i(2,p)\}_{i=76}^{\infty}$ is periodic with period five, with all values nonzero.
Proof. See [3].

Now, consider the following model. Suppose that we have a state space \( \mathcal{X} = \{1, 2, \ldots, n\} \), a frequency space \( \mathcal{T} \) and functions \( S : \mathcal{X} \rightarrow \mathcal{T} \) (the send function) and \( R : \mathcal{X} \rightarrow \mathcal{T} \) (the receive function) such that for \( i, j \in \mathcal{X} \), \( R(i) \neq S(i) \) and \( i \) is directly accessible from \( j \) if and only if \( 0 < i - j \leq k \) and \( R(i) = S(j) \).

To see where such a model might arise, consider the set \( \mathcal{T} = \{0, 1\} \), i.e. \( (S(i), R(i)) \in \{(0, 1), (1, 0)\} \) for \( 1 \leq i \leq n \). The value \( (S(j), R(j)) \) can be interpreted as the orientation of a receiver-transmitter repeater station at location \( j \) with transmission at frequency \( S(j) \) and reception at frequency \( R(j) \). Reinterpreted, this case also covers the scenario of Example 1, above. In that situation \( k = 5 \).

To establish the connection between recurrence bounds and paths, let \( \Gamma(S, R, k, n) \) be the set of all allowable paths from state 1 to state \( n \) with all hops of size \( k \) or less\(^1\) (see Example 2, below for an illustration). For a path \( \gamma \in \Gamma(S, R, k, n) \), through states \( i_1, i_2, \ldots, i_m \), define the associated set of hops

\[
H(\gamma) \overset{\text{def}}{=} \{i_{j+1} - i_j : 1 \leq j \leq m - 1\}.
\] (4)

Finally, let \( E(S, R, k, n, p) = 0 \) for \( n \leq 0, E(S, R, k, 1, p) = p \) and

\[
E(S, R, k, n, p) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma(S, R, k, n)} p^{l(\gamma)} = \sum_{\gamma \in \Gamma(S, R, \infty, n)} \left( \prod_{z \in H(\gamma)} I_{[z \leq k]} \right) p^{l(\gamma)},
\] (5)

for \( n \geq 1 \), where

\[
I_{[A]} = \begin{cases} 1, & \text{if } A \text{ holds} \\ 0, & \text{otherwise} \end{cases}
\] (6)

and \( l(\gamma) \) is the path length of \( \gamma \), i.e., the number of states visited in \( \gamma \).

We shall refer to \( E(S, R, k, n, p) \) as the output signal of the system. Note that when \( p = 1 \), \( E(S, R, k, n, p) \) is simply \( \|\Gamma(S, R, k, n)\| \), i.e. the number of paths in \( \Gamma(S, R, k, n) \).

We demonstrate the scenario with the following example.

**Example 2.** Consider \( \mathcal{T} = \{+, -\} \) and \( \|\mathcal{X}\| = 8 \), and note that in this case \( R \) is determined by \( S \), since \( \|\mathcal{T}\| = 2 \). Figure 1 presents three possible paths through \( \mathcal{X} \), for the pattern \( \{S(i)\} = \{+, -, +, -, +, -, +, -\} \). If

\(^1\)We refer to a direct move from state \( j \) to state \( i \) as a "hop" of size \( i - j \).
we designate a path via the vector of $\mathcal{X}$-states visited, then it may be verified that in this case, $\Gamma(S,R,2,8) = \{(1,2,4,6,7,8), (1,3,4,6,7,8), (1,3,5,6,7,8)\}$ (i.e. the set of paths given in Figure 1), while

$$\Gamma(S,R,\infty,8) = \Gamma(S,R,2,8) \cup \{(1,2,5,6,7,8), (1,3,7,8), (1,2,7,8), (1,6,7,8), (1,2,5,8), (1,3,5,8), (1,3,4,8), (1,2,4,8), (1,8)\}. \quad (7)$$

A method for determining optimal repeater configurations which maximize output signal could be of interest. As indicated by the preceding example, the process of exhaustively listing all paths could be a somewhat daunting and cumbersome task for large $||\mathcal{X}||$.

The recursive nature of path construction for given $(\mathcal{X},S,R)$ suggests a close connection with recurrences. In fact, suppose that $\{b_i\}$ and $\{\alpha_{i,j}\}$ satisfy (1) and (2) with $b_1 \in \{-1,1\}$.

Define $S$ and $R$ by

$$S(j) = \begin{cases} 1, & b_j \geq 0, \\ 0, & \text{otherwise}, \end{cases} \quad (8)$$

and $R(j) = 1 - S(j)$ for $j \geq 1$, and set $B_n(S,k,p)$, recursively in $n$, from $S$ via $B_i(S,k,p) = 0$ for $i \leq 0$, $B_1(S,k,p) = b_1$ and
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\[ B_n(S, k, p) = -p \sum_{\substack{1 \leq i \leq n-1 \\ S(i) \neq S(n)}} B_i(S, k, p), \]  \hspace{1cm} (9)

for \( n \geq 2. \)

It follows easily that \(|b_i| \leq |B_i(S, k, p)|\), for all \( i \geq 1 \) (see [2], Lemma 1).

As is shown by the following lemma, paths through \( \mathcal{X} \) and values of linear recurrences of the type mentioned above are, indeed, simply and intimately connected.

**Lemma 2.** We have

\[ E(S, R, k, n, p) = p \ |B_n(S, k, p)| \]  \hspace{1cm} (10)

for all \( n \geq 1. \)

In particular, the polynomial \( p \cdot |B_n(S, k, p)| \) is the generating function for the number of paths of given length \( i \) in \( \Gamma(S, R, k, n) \).

**Proof.** The proof follows by induction on \( n. \) The case for \( n = 1 \) is trivial. Now assume the result holds for \( n < N. \) For \( \gamma \in \Gamma(S, R, k, n) \), conditioning on the last state visited in \( \gamma \) leads to

\[
E(S, R, N, k, p) = \sum_{1 \leq i \leq k} \sum_{\gamma \in \Gamma(S, R, k, N-i) \mid S(N-i) = R(N)} p \cdot |\gamma| \\
= \sum_{1 \leq i \leq k} \sum_{\gamma \in \Gamma(S, R, k, N-i) \mid S(N-i) \neq S(N)} p \cdot E(S, R, N-i, k, p). \]  \hspace{1cm} (11)

Employing (11) and the induction hypothesis gives,

\[
E(S, R, N, k, p) = \sum_{1 \leq i \leq k} \sum_{S(N-i) \neq S(N)} p^2 \ |B_{N-i}(S, k, p)| \\
= p \sum_{S(N-i) \neq S(N)} p \cdot |B_{N-i}(S, k, p)| \\
= p \ |B_N(S, k, p)|. \]  \hspace{1cm} (12)
This completes the proof of the equality in (10). The assertion regarding the generating function for the number of paths of given length, then follows from (5).

Now, for $p > 0$ and $k > 1$, let

$$V_{n,\mathcal{T}}(k, p) \overset{\text{def}}{=} \max_{S, R} \{ E(S, R, k, n, p) \}. \quad (13)$$

Setting $\mathcal{T} = \{0, 1\}$, and taking maximums of both sides of (10) in Lemma 2, leads directly to the following.

**Theorem 3.** Suppose $p > 0$ and $k > 1$, then for all $n \geq 1$,

$$V_{n,\{0,1\}}(k, p) = U_n(k, p). \quad (14)$$

**Remark.** In [2] the values of $\{U_n(k, 1)\}$ were studied in the context of an extension of a theorem of Graham and Sloane [8] regarding inverses of $\{0, 1\}$ matrices (see Theorem 5, below). From Lemma 2, the values obtained may be interpreted as the maximal number of paths for any of the $2^n$ possible sign configurations of $\{b_1, b_2, \ldots, b_n\}$. Questions still remain regarding the values of $U_n(k, 1)$ for even $k$ (see [2] for further discussion).

We remark that Lemma 2 also has theoretical value in regard to dealing with linear recurrences.

As mentioned in the following remark, our considerations may be extended to higher dimensional situations. Work has begun on a thorough investigation of the types of recurrences arising there.

**Remark.** (Extensions to higher dimensions) Consider the two dimensional recurrence

$$b_{n,m} = \sum_{i \leq n} \sum_{j \leq m} \beta_{n, m, i, j} b_{i, j}, \quad (15)$$

where $b_{i, j} = 0$ when $\min\{i, j\} \leq 0$, and

$$\beta_{n, m, i, j} \in [-p, 0], \quad (16)$$

for all $(i, j)$. Without loss of generality we will assume that $b_{1, 1} \in \{-1, 1\}$. 
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Now, define

\[ U_{n,m}(p) \overset{\text{def}}{=} \max \{|b_{n,m}| : \{b_{i,j}\}, \{\beta_{n,m,i,j}\} \text{ satisfy (15) and (16)}\}, \]

for \( n,m \geq 1 \).

As in the one-dimensional case, connections between optimization for recurrences as in (15) and maximal paths across lattices are possible. Similarly, extensions to higher dimensional lattices could also be of interest.

We now turn to some further discussion of the application at hand.

2. An Application to Directed Transmissions Through Repeaters

As earlier, consider the set \( T = \{0, 1\} \), i.e. \((S(i), R(i)) \in \{(0, 1), (1, 0)\}\) for \( 1 \leq i \leq n \). The value \((S(j), R(j))\) can be interpreted as the orientation of a receiver-transmitter repeater station at location \( j \) with transmission at frequency \( S(j) \) and reception at frequency \( R(j) \). In a system where communication lines are unreliable, it may be of interest to maximize the number of possible paths that information may take within the system. Maximal transmission distance can be interpreted as the value of \( k \), and \( p \) may represent an amplification (\( p \geq 1 \)) or loss of signal (\( p < 1 \)) when passing through station \( j \). For our purposes, here, we will assume, for simplicity that attenuation of signal is minimal (within the maximal transmission distance, \( k \)), and that any potential delays and multipath interference may be dealt with.

Applying Theorem 1 in this setup, we have that regardless of station orientations, if \( k = 2 \) (i.e. the only stations reachable are those at a distance of either one or two) and \( p < (1/3)^{(1/3)} \), then as the length of the system, \( n \), tends to infinity, the system output will necessarily tend to zero. On the other hand, if \( p > (1/3)^{(1/3)} \), then as the length of the system tends to infinity, there exist configurations for which the output increases without bound.

In the infinite order case (\( k = \infty \)), the following was recently obtained.
Theorem 4. Suppose that $p > 0$ and $m = [1/p]$. Then,

$$U_n(\infty, p) = \begin{cases} 
  p, & \text{if } n = 2, \\
  \max(p, p^2), & \text{if } n = 3, \\
  \left[ \frac{n-2}{2} \right] \left[ \frac{n-1}{2} \right] p^3 + p, & \text{if } 4 \leq n \leq 2m + 2, \tag{17} \\
  (n-2)p^2, & \text{if } n = 2m + 3, \\
  pU_n-1(\infty, p) + U_n-2(\infty, p), & \text{if } n \geq 2m + 4.
\end{cases}$$

Proof. See [5].

Theorem 4 provides easily computable signal bounds. The optimal configurations are provided in Table 5 of [5]. In this case, regardless of the value of $p > 0$, the optimal output signal tends to infinity with $n$.

As a final example, we mention the following for the case $p = 1$ and odd $k$.

**Theorem 5.** Suppose $k \geq 1$ is odd, and define $\{Z_i(k)\}$ by $Z_i(k) \equiv 0$ for $i \leq 0$, $Z_1(k) = 1$, and for $j \geq 2$,

$$Z_j(k) = \sum_{i=0}^{[k/2]} Z_{j-2i-1}(k) = Z_{j-1}(k) + Z_{j-3}(k) + \cdots + Z_{j-k}(k). \tag{18}$$

Then, $U_n(k, 1) = Z_n(k)$ for all $n \geq 2$.

Proof. See [2].

Applying Theorems 3 and 5 leads to the following.

**Example 1 (revisited).** In the context of Example 1, suppose that $n$ slabs are laid across the room. Applying Theorem 3 and 5, and considering the structure of $\{Z_i(5)\}$, we obtain that the maximal number of paths is obtained by alternating "left" and "right" slabs. What is perhaps most interesting is that this is not the case for $k = 4$ or $k = 6$.

**Remark.** Note that several generalizations of the networks and path systems considered here are possible. For instance one may consider different
sets $T$, (which allow for additional frequencies), allow $p$ to vary by station, or incorporate a diminishing signal with distance. As mentioned earlier, generalization to higher dimensions than that of the simpler serial model discussed here, could also be of interest. Probabilistic and graph-theoretic considerations also appear to be interesting avenues for further inquiry.

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References


