Remarks on linear recurrences of the form $y_n = y_{n-1} + a_{n-1}y_{n-2}$

Kenneth S. Berenhaut$^1$, Augustine B. O'Keefe, Filip Saidak
Department of Mathematics, Wake Forest University
Winston-Salem, NC 27109

Abstract
We study recursive sequences of the form $y_n = y_{n-1} + a_{n-1}y_{n-2}$
with positive integral coefficients. Several inequalities for solutions
are proven and some results regarding fixed sums are obtained.

Key words: Recursive sequence, Second order, Partitions, Continued fractions.
2000 MSC: 39A10, 11B37, 11P81, 05A17.

1 Introduction
This paper studies general linear recurrences of the form

$$y_n = y_{n-1} + a_{n-1}y_{n-2} \quad (n \geq 1),$$

with $a_i \in \mathbb{N}$ for $i \geq 1$, $y_{-1} = 0$ and $y_0 = 1$. These are generalizations of the Fibonacci sequence, which has been researched extensively in the past. Our
main aim here is to present a collection of results stemming from our study
of the structure of solutions to (1). This work complements the research
done recently in our paper [1], where we investigated the related recurrence

$$y_n = a_ny_{n-1} + y_{n-2} \quad (n \geq 1).$$

In [1] we proved the following two theorems concerning maximality of
values of $y_k$ of the recurrent sequence (2):

Theorem 1. Suppose that we have $\sum_{i=1}^k a_i = N$, in recurrence (2). Then

$$y_k(a_1, a_2, \ldots, a_k) \leq y_k(w, x, x, \ldots, x, w, w, \ldots, w),$$

where $x = \lfloor N/k \rfloor$ and $w = x + 1$. Here $\lfloor s \rfloor$ denotes the integer part of a
real number $s$. Also note that the number of $w$'s in (3) is precisely $N - kx$,
while the number of $x$'s is $k(x + 1) - N$.

Theorem 2. Suppose that, in the recurrence in (2), we have $\sum_{i=1}^{k+1} a_i = N + M$ and max $a_i = M$. Then

$$y_{k+1}(a_1, a_2, \ldots, a_{k+1}) \leq y_{n+1}(M, 1, 1, \ldots, 1).$$

AMS Classification: 11A25

---

$^1$The first author acknowledges financial support from an Archie Fund grant.

$^2$Email addresses: berenhka@wfu.edu (Kenneth S. Berenhaut), okeefeb@wfu.edu (Augustine B. O'Keefe), saidakf@wfu.edu (Filip Saidak).

CONGRESSUS NUMERANTUM 200 (2000), pp. 141-151
From this point forward, we will look at properties of the recurrence in (1), and use notation corresponding to it. We begin by noting that:

**Theorem 3.** We have

$$y_i = \sum_{S \subseteq \{1, 2, \ldots, i-1\}} \prod_{j \in S} a_j,$$  \hfill (5)

where the sum is over all sets $S \subseteq \{1, 2, \ldots, i-1\}$ such that $i, j \in S$ with $i < j$ implies $j - i \geq 2$.

**Proof.** The result follows via a straightforward induction.

It turns out that despite the apparent similarities between the two recurrences, the situation is much more complicated for the recurrence in (1). In fact, it needs to be said that thus far we have been unable to prove the analog of Theorems 1 and 2. Instead, in Section 3 below, we settle for formulating a conjecture that describes the situation in an exact manner. In face of the difficulties arising in this new setting, the aim of the paper will be twofold:

- collect results that characterize the structure of solutions of (1),
- explicitly describe connections between the solutions of (1) and the theory of continued fractions.

Let us mention that in the theory of continued fractions a very important role is played by the canonical representations of finite segments of continued fractions. If $\gamma = [\gamma_0; \gamma_1, \gamma_2, \gamma_3, \ldots]$ is a given continued fraction, then the $k$-th order convergent of $\gamma$ is given by

$$c_k = [\gamma_0; \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k] = \frac{p_k}{q_k},$$  \hfill (6)

and these finite fractions serve as best approximations to $\gamma$. One of the fundamental properties of the $k$-th order convergents (see Hardy and Wright [2] or Khinchin [3]) is the relation:

$$\frac{p_k}{q_k} = \frac{\gamma_0 p_{k-1} + p_{k-2}}{\gamma_0 q_{k-1} + q_{k-2}} \quad (k \geq 2).$$  \hfill (7)

Here the numerators and denominators both satisfy the linear recurrence studied in [1]. The linear recurrence in (1) is also closely connected to continued fraction convergents (see Theorem 4, Corollary 1 and Lemma 4, below). In fact, the connection here is even more explicit and useful in applications.

The paper proceeds as follows. In Section 2 we introduce some preliminary results and lemmas regarding properties of $\{y_k(a_1, a_2, \ldots, a_{k-1})\}$ while in Section 3 we discuss some results related to some fixed sum problems.

### 2 Preliminary results and lemmas

This section contains several preliminary results and lemmas regarding properties of $\{y_k(a_1, a_2, \ldots, a_{k-1})\}$. For ease of notation we will use the vector $(a_1, a_2, \ldots, a_{k-1})$ in place of $y_k(a_1, a_2, \ldots, a_{k-1})$. Also, let us denote $(a_{m,k}) = (a_m, \ldots, a_k)$ and $(b^k) = (b, \ldots, b)$ when the number of $b$'s is $k$. For convenience, in keeping with (\ref{eq:7}), we take $(a_{t+1,1}) = (a_{t+2,1}) = () = 1$ and $(a_{t+1,t+1}) = 0$ for $t > 2$. Some of the most basic properties of operations on $k$-tuples are given in the following lemma.

**Lemma 1.** The following statements hold.

1. For all $b, c \geq 1$, $(c) = c + 1$, and $(b, c) = b + c + 1$. For all $i \leq j$, we have

$$\alpha_{ij} = \alpha_{i,j-1} + \alpha_{j-1,j}.$$  \hfill (8)

2. (Splitting) For all $(a_{1,k})$ and $1 \leq m \leq k - 1$,

$$\alpha_{1,k} = \alpha_{1,m} \alpha_{m+2,k} + \alpha_{1,m-1} \alpha_{m+1,k}.$$  \hfill (9)

3. (Symmetry) For all $(a_{1,k})$,

$$\alpha_{1,k} = (a_0, a_1, \ldots, a_{k-2}, a_k) \overset{\text{def}}{=} \alpha_{1,k}.$$  \hfill (10)

4. (Monotonicity) The function

$$y_k(x_1, x_2, \ldots, x_{k-1}) = (x_1, x_2, \ldots, x_{k-1})$$  \hfill (11)

is monotonous in each variable $a_i$, and

$$y_{k+1}(x_1, x_2, \ldots, x_{k-1}, x) > y_k(x_1, x_2, \ldots, x_{k-1}),$$  \hfill (12)

for all $x \geq 1$.

**Proof.** The equalities in (1) and (2) follow directly from (1), as does (5). To see (3), note that, for $k = 2$,

$$(a_{1,2}) = (a_0, a_2) = a_1 + a_2 + 1 = (a_{1,1})(a_{3,2}) + (a_{1,2})a_2(a_{4,2}).$$  \hfill (13)

Hence, from (2) and induction, we have

$$\begin{align*}
(a_{1,k}) &= (a_{1,k-1}) + a_k(a_{1,k-2}) \\
&= [(a_{1,m})(a_{m+2,k-1}) + (a_{1,m-1})(a_{m+1,k-1})] \\
&\quad + a_k[(a_{1,m})(a_{m+2,k-2}) + (a_{1,m-1})(a_{m+1,k-2})] \\
&= (a_{1,m})(a_{m+2,k-1}) + a_k(a_{m+1,k-2}) \\
&\quad + (a_{1,m-1})(a_{m+1,k-1}) + a_k(a_{m+1,k-2}) \\
&= (a_{1,m})(a_{m+2,k}) + (a_{1,m-1})(a_{m+1,k}).
\end{align*}$$  \hfill (14)
The symmetry in (4), is most easily seen by applying the result in Theorem 3 regarding index-gaps.

We will refer to the splitting in (9) as “splitting \((a_{1,k})\) about \(a_{n+1}\)”. The following two lemmas will be crucial for understanding the structure of optimal \((a_i)\), with respect to the value of \((a_i)\) under fixed sum \(\sum a_i\) constraints.

Lemma 2. For all positive integers \(k\) and \(b, c \geq 1\)

\[
(a_{1,k}, b, c, a_{k+1,n}) \leq (a_{1,k}, b - 1, c + 1, a_{k+1,n}),
\]

whenever

\[
(a_{1,k})(a_{k+2,n}) \geq (a_{1,k-1})(a_{k+1,n}).
\]

**Proof.** By splitting, we have

\[
(a_{1,k}, x, y, a_{k+1,n}) = (a_{1,k})(y, a_{k+1,n}) + (a_{1,k-1})x(a_{k+1,n})
\]

\[
= (a_{1,k})(a_{k+1,n} + y(a_{k+2,n})]
\]

\[
+ (a_{1,k-1})x(a_{k+1,n}).
\]

Hence applying (17) for \((x, y) = (b - 1, c + 1)\) and \((x, y) = (b, c)\), and taking the difference, we have

\[
(a_{1,k}, b - 1, c + 1, a_{k+1,n}) - (a_{1,k}, b, c, a_{k+1,n})
\]

\[
= (a_{1,k})(a_{k+2,n}) - (a_{1,k-1})(a_{k+1,n}),
\]

and the lemma follows.

Similarly, setting \(n = k\) in (15), and noting that \((a_{n+2,n}) = (a_{n+1,n}) = 1\) and \((a_{1,n}) > (a_{1,n-1})\), we have the following lemma.

Lemma 3. For all positive integers \(k\) and \(b, c \geq 1\)

\[
(a_{1,k}, b, c) < (a_{1,k}, b - 1, c + 1).
\]

Next we point out an important connection between recurrences satisfying (1) and continued fraction convergents.

Theorem 4. We have

\[
(a_{1,k}) = [1; a_{1}, a_{2}, \ldots, a_{k}]\frac{a_{2}}{(a_{2}, k)}
\]

where

\[
[1; a_{1}, a_{2}, \ldots, a_{k}] = 1 + \frac{a_{1}}{1 + \frac{a_{2}}{1 + \frac{a_{3}}{1 + \frac{a_{4}}{1 + \ddots}}}}
\]

is the associated continued fraction convergent.

**Proof.** Note that

\[
\frac{(a_{1,k})}{(a_{2,k})} = \frac{(a_{2,k}) + a_{1}(a_{3,k})}{(a_{2,k})} = 1 + \frac{a_{1}(a_{3,k})}{(a_{2,k})} = 1 + \frac{a_{1}}{(a_{2,k})/(a_{3,k})}.
\]

The result now follows by induction.

The following corollary follows from repeated application of Theorem 4.

Corollary 1. We have

\[
(a_{1,k}) = [1; a_{1}, a_{2}, \ldots, a_{k}] \cdot [1; a_{2}, \ldots, a_{k}] \cdots [1; a_{k}].
\]

In view of Corollary 1, study of solutions of (1) can be viewed in terms of iterated products of continued fraction convergents.

For the remainder of this section we consider various inequalities regarding equation (1). The first of these inequalities is a result of the expression in (20) and simple properties of continued fractions.

Lemma 4. For \(a_0 = 1\) and any \(i, j \geq 1\),

\[
[a_0; a_1, a_2, \ldots, a_i](a_{j,k}) \leq (a_{1,k})\leq [a_0; a_1, a_2, \ldots, a_{j-1}](a_{j,k}).
\]

Lemma 5. For any \(x \geq 1\)

\[
(x + 1)(a_{1,k}) < (a_{1,k}, 1, x) < (x + 2)(a_{1,k}).
\]

**Proof.** The result follows since

\[
(a_{1,k}, 1, x) = (a_{1,k} + (a_{1,k})x
\]

\[
= (a_{1,k}) + (a_{1,k} + (a_{1,k} + x)
\]

\[
= x(1 + (a_{1,k}) + (a_{1,k} + 1).
\]
Lemma 6. For all $x \geq 1$,
\[
(a_{1,x}, 1, x) > \frac{x+1}{2} (a_{1,x}, 1).
\] (26)

Proof. Here, by the LHS inequality in (24), $(a_{1,x}, 1, x) > (x + 1)(a_{1,x})$, while
\[
(a_{1,x}) = (a_{1,x}) + (a_{1,x-1}) < 2(a_{1,x}).
\] Hence
\[
\frac{(a_{1,x}, 1, x)}{(a_{1,x})} > \frac{x+1}{2} (a_{1,x}). \Box
\] (27)

In the next section, we restrict attention to the question of maximizing $y_{i+1}(a_1, a_2, \ldots, a_i)$ for $a_1, a_2, \ldots, a_i \geq 1$, under the fixed sum constraint $a_1 + a_2 + \cdots + a_i = S$ for some fixed $S \geq 1$.

3 Maximization under fixed sum constraints

To further study properties of solutions of (1), let us introduce the following notation: for $k \geq 1$ let
\[
([a, b]^k) = (a, b, a, b, \ldots, a, b)
\] (29)
where the right hand side of the equation is of length $2k$.

The following analogue of Theorem 1 is strongly suggested by computations and some partial theoretical results.

Conjecture 1. For $k \geq 1$, and given $(a_1, a_2, \ldots, a_k)$, let $N = \sum_{i=1}^{k} a_i$, $n_1 = \lfloor \frac{N}{2} \rfloor$, $n_2 = \lceil \frac{N}{2} \rceil$, $x = \lceil \frac{N-1}{y} \rceil$, $y = \lceil \frac{N+1}{y} \rceil$, and $n_e = (N - n_1) - n_2$. Then, for $k$ odd,
\[
(a_1, a_2, \ldots, a_k) \leq ([x, x]^k, y, 1, \ldots, y, 1, y, \ldots, 1, y, [x, x]^k),
\] (30)
where $k_1 = \lfloor \frac{k+1}{2} \rfloor$ and $k_2 = \lceil \frac{k+1}{2} \rceil$. Note that $n_e$ then denotes the number of $x$'s and so the number of $y$'s is $n_0 = \lceil \frac{k}{2} \rceil - n_e - n_2$. For even $k$, we have
\[
(a_1, a_2, \ldots, a_k) \leq ([x, x]^k, [y, y]^j, 1, y, [y, y]^j, 1, x, [x, x]^k),
\]
where $j_1 = \lfloor \frac{k}{2} \rfloor$ and $j_2 = \lceil \frac{k}{2} \rceil$. \Box

The following lemma provides some intuition towards understanding the alternating structure implied by Conjecture 1.

Lemma 7. Suppose $\sum a_j = S$, then there exist $b_1, \ldots, b_i$ with $\sum b_j = S$ and
\[
(a_1, a_2, \ldots, a_i) \leq (b_1, b_2, \ldots, b_i),
\] (32)
where each $b_j > 1$ is "isolated" in the sense that $b_j > 1$ implies either (i) $b_{j-1} = b_{j+1} = 1$ or (ii) $j = i$ and $b_{i-1} = 1$ or (iii) $j = 1$ and $b_2 = 1$.

Proof. The result follows directly from Lemmas 2 and 3 and symmetry. To see this, note that (16) does not depend on $(b, c)$. From Lemma 2, we have
\[
(b_1, 1, a_2, a_3, \ldots, a_k) \geq (a_1, a_2, a_3, \ldots, a_k),
\] (33)
where $b_1 = a_1 + 2$. Hence, suppose that $a_2 = 1$ and that there is some $j \leq i - 1$ such that $\min(a_j, a_{j+1}) > 1$, and let $J$ be the smallest such $j$, i.e. $J = \min\{j : a_j > 1 \text{ and } a_{j+1} > 1\}$. Then, by Lemma 2, either
\[
(a_1, a_2, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) \leq (a_1, a_2, \ldots, a_{j-1}, b_j, 1, a_{j+2}, \ldots, a_k),
\]
where $b_j = a_j + 1$, and the result follows by induction. \Box

The next several lemmas provide a basis for the "levelness" and symmetry implied by Conjecture 1.

Lemma 8. For all integers $k > 0$ and $b \geq 3$ and $c \geq 1$ satisfying $b-c-1 \geq 1$,
\[
(b - 1, 1, a_1, k, 1, c + 1) > (b, 1, a_1, k, 1, c + 1).
\] (34)

Proof. Similar to the proof of Lemma 2, we have
\[
(x, 1, a_1, k, 1, y) = (x)(a_1, k, 1, y) + (a_2, k, 1, y)
= (x)(a_1, k, 1, y) + (a_1, k, 1, y)
+ (y)(a_2, k, 1, y) + (a_2, k, 1, y)
= (a_2, k, 1, y) + (x)(a_1, k, 1, y) + (y)(a_2, k, 1, y) + (x)(a_1, k, 1, y).
\] (35)
Hence applying (35) for \((x, y) = (b-1, c+1)\) and \((x, y) = (b, c)\), and taking the difference, we obtain
\[
(b-1, 1, \alpha_{1,k}, 1, c+1) - (b, 1, \alpha_{1,k}, 1, c) \\
= (\alpha_{2,k}) - (\alpha_{1,k-1}) + [b - c - 1](\alpha_{1,k}).
\]  
(36)

The result then follows since \((\alpha_{1,k}) > (\alpha_{1,k-1})\).

**Lemma 9.** For all positive integers \(k, n\), with \(n \geq k + 1\), and \(b \geq 3\) and \(c \geq 1\) satisfying \(b - c - 1 \geq 1\), we have
\[(\alpha_{1,k}, 1, b - 1, 1, c + 1, 1, \alpha_{k+1,n}) > (\alpha_{1,k}, 1, b, 1, c, 1, \alpha_{k+1,n}).\]
(37)

**Proof.** First, splitting about \(b\) and then \(c\), we have
\[
(\alpha_{1,k}, 1, b, 1, c, 1, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1, c, 1, \alpha_{k+1,n}) + b(\alpha_{1,k})(c, 1, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1, \alpha_{k+1,n}) + c(\alpha_{1,k}, 1)(\alpha_{k+1,n}) \\
+ b(\alpha_{1,k})(1, \alpha_{k+1,n}) + bc(\alpha_{1,k})(\alpha_{k+1,n}).
\]  
(38)

Similarly, we have
\[
(\alpha_{1,k}, 1, b - 1, 1, c + 1, 1, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1, c + 1, 1, \alpha_{k+1,n}) + [b - 1](\alpha_{1,k})(c + 1, 1, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1, \alpha_{k+1,n}) + [c + 1](\alpha_{1,k}, 1)(\alpha_{k+1,n}) \\
+ [b - 1](\alpha_{1,k})(1, \alpha_{k+1,n}) + [b - 1][c + 1](\alpha_{1,k})(\alpha_{k+1,n}).
\]  
(39)

Taking the difference we obtain
\[
(\alpha_{1,k}, 1, b - 1, 1, c + 1, 1, \alpha_{k+1,n}) - (\alpha_{1,k}, 1, b, 1, c, 1, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(\alpha_{k+1,n}) - (\alpha_{1,k}, 1)(\alpha_{k+1,n}) \\
+ [b - c - 1](\alpha_{1,k})(\alpha_{k+1,n}) \\
= (\alpha_{1,k-1})(\alpha_{k+1,n}) - (\alpha_{1,k})(\alpha_{k+2,n}) \\
+ [b - c - 1](\alpha_{1,k})(\alpha_{k+1,n}).\]
(40)

The result then follows since \((\alpha_{k+1,n}) > (\alpha_{k+2,n}).\)

**Lemma 10.** For all positive integers \(k, n\), with \(n \geq k + 1\), and \(x \geq 2\) and \(y, z \geq 1\) satisfying \(x - y - 1 \geq z\), we have
\[(\alpha_{1,k}, 1, x - 1, 1, y + 1, 1, \alpha_{k+1,n}) > (\alpha_{1,k}, 1, x, 1, y, z, \alpha_{k+1,n}).\]
(41)

**Proof.** Here, we have
\[
(\alpha_{1,k}, 1, x, 1, y, z, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1, y, z, \alpha_{k+1,n}) + (\alpha_{1,k})(y, z, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1)(x, \alpha_{k+1,n}) + (\alpha_{1,k}, 1)(\alpha_{k+1,n})y \\
+ (\alpha_{1,k})(x, \alpha_{k+1,n})z + (\alpha_{1,k})(\alpha_{k+1,n})x.
\]  
(42)

We wish to compare (42) with
\[
(\alpha_{1,k}, 1, x - 1, 1, y + 1, z, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1, y + 1, z, \alpha_{k+1,n}) + (\alpha_{1,k})(x - 1)(y + 1, z, \alpha_{k+1,n}) \\
= (\alpha_{1,k}, 1)(1)(x, \alpha_{k+1,n}) + (\alpha_{1,k}, 1)(\alpha_{k+1,n})y + [1] \\
+ (\alpha_{1,k})(x, \alpha_{k+1,n})z + (\alpha_{1,k})(\alpha_{k+1,n})x + y - 1, y - 1).
\]  
(43)

Thus, taking the difference of (43) and (42), we obtain
\[
(\alpha_{1,k}, 1)(\alpha_{k+1,n}) - (\alpha_{1,k})(x, \alpha_{k+1,n}) \\
+ (\alpha_{1,k})(\alpha_{k+1,n})z - (\alpha_{1,k})(x, \alpha_{k+1,n})z \\
= (\alpha_{1,k-1})(\alpha_{k+1,n}) - (\alpha_{1,k})(\alpha_{k+2,n}) \\
+ (\alpha_{1,k})(\alpha_{k+1,n})z - (\alpha_{1,k})(\alpha_{k+1,n})z.
\]  
(44)

And the result follows, since \((\alpha_{k+1,n}) > (\alpha_{k+2,n}).\)

**Remark 1.** Note that if \(y = 1\) in (41), we have
\[(\alpha_{1,k}, 1, x - 1, 1, 2, 1, \alpha_{k+1,n}) > (\alpha_{1,k}, 1, x, 1, 1, z, \alpha_{k+1,n}).\]
(45)

and hence, utilizing Lemma 2, either
\[(\alpha_{1,k}, 1, x - 1, 1, 1, z + 1, \alpha_{k+1,n}) > (\alpha_{1,k}, 1, x, 1, 1, z, \alpha_{k+1,n}).\]
(46)

or
\[(\alpha_{1,k}, 1, x - 1, 1, z + 1, 1, \alpha_{k+1,n}) > (\alpha_{1,k}, 1, x, 1, 1, z, \alpha_{k+1,n}).\]
(47)
Next, we give the following general inequality for arbitrary values $b$ and $c$.

**Lemma 11.** For all positive integers $k, m, n$, with $n \geq m + 1 \geq k + 2$, and $b \geq 5$ and $c \geq 1$ satisfying $b - c - 1 \geq 3$, we have

\[
(a_{1,k}, 1, b - 1, 1, a_{k+1,m}, 1, c + 1, a_{m+1,n}) > (a_{1,k}, 1, b, 1, a_{k+1,m}, 1, c, 1, a_{m+1,n}).
\]  

(48)

**Proof.** Splitting the right hand side of the inequality, we obtain

\[
(a_{1,k}, 1, b, 1, a_{k+1,m}, 1, c, 1, a_{m+1,n}) = (a_{1,k}, 1)(1, a_{k+1,m}, 1, c, 1, a_{m+1,n}) + (a_{1,k})b(a_{k+1,m}, 1, c, 1, a_{m+1,n})
\]

\[
= (a_{1,k}, 1)(1, a_{k+1,m}, 1, c, 1, a_{m+1,n}) + (a_{1,k})(1, a_{k+1,m})(a_{m+1,n})c
\]

\[
+ (a_{1,k})(a_{k+1,m}, 1)(1, a_{m+1,n})b
\]

\[
+ (a_{1,k})(a_{k+1,m})(a_{m+1,n})bc.
\]  

(49)

Similarly, upon splitting the left hand side in (48), taking the difference and employing (27) we have

\[
(a_{1,k}, 1)(1, a_{k+1,m})(a_{m+1,n}) - (a_{1,k})(a_{k+1,m}, 1)(a_{m+1,n})
\]

\[
+ (a_{1,k})(a_{k+1,m})(a_{m+1,n})[b - c - 1]
\]

\[
> (a_{1,k})(a_{k+1,m})(a_{m+1,n}) - 4(a_{1,k})(a_{k+1,m})(a_{m+1,n})
\]

\[
+ (a_{1,k})(a_{k+1,m})(a_{m+1,n})[b - c - 1]
\]

\[
= (a_{1,k})(a_{k+1,m})(a_{m+1,n})[b - c - 4].
\]  

(50)

Thus, the result follows if $b - c \geq 4$. \qed

Compiling the results from the preceding lemmas, we have a proof of the following properties implied by Conjecture 1.

**Theorem 5.** For given $(a_1, a_2, \ldots, a_k)$, satisfying $N = \sum_{i=1}^{k} a_i$, there exists $(b_1, b_2, \ldots, b_k)$ satisfying

\[
y_{k+1}(a_1, a_2, \ldots, a_k) \leq y_{k+1}(b_1, b_2, \ldots, b_k),
\]  

(51)

and the following four properties:

\[
lk \sum b_j = N. \quad \text{Each } b_j > 1 \text{ is isolated in the sense of Lemma 7. If } b_j > 1, \text{ and } b_i = 1 \text{ for } j < i < l \text{ then } |b_j - b_i| \leq 1. \quad \text{If } b_j > 1 \text{ and } b_i > 1, \text{ then } |b_j - b_i| \leq 3.
\]