Periodicity and boundedness for the integer solutions to a minimum-delay difference equation

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In this paper, we study periodicity and boundedness for the integer solutions to a minimum-delay difference equations. As an application, a recent theorem regarding absolute-difference equations is extended.

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1. Introduction

In this paper, we study periodicity and boundedness of integer solutions to minimum-delay difference equations of the form

\[ y_n = \min\{y_{n-k_1}, y_{n-m_1}, y_{n-m_2}, \ldots, y_{n-k_L}, y_{n-m_L}\}, \quad n \geq 0, \quad (1) \]

with delay matrix

\[ K = \begin{bmatrix}
k_1 & m_1 \\
k_2 & m_2 \\
& \\
& \\
k_L & m_L
\end{bmatrix}, \quad (2)\]

where \( k_i, m_i \geq 1 \) for \( 1 \leq i \leq L \) and \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in \mathbb{Z} \), with

\[ s = \max\{k_1, m_1, k_2, m_2, \ldots, k_L, m_L\}. \quad (3)\]

We will place particular attention on the \( L = 2 \) case in (1), i.e. on the equation

\[ y_n = \min\{y_{n-k_1}, y_{n-m_1}, y_{n-k_2}, y_{n-m_2}\}, \quad n \geq 0, \quad (4)\]

and be interested in how behaviour of solutions vary with the associated \( 2 \times 2 \) delay matrix. To address how behaviour varies with \( L \) in (1), we will also briefly consider the (linear) \( L = 1 \) case of solutions to \( y_n = y_{n-k} - y_{n-m} \) for given \( K = [k, m] \). For equations of

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the form \( y_n = g(y_{n-k}y_{n-m}) \), for more general functions, \( g \), see for instance [24] and the references therein.

Behaviour of minimum (and maximum) difference equations have been studied by several authors in the recent past (see Refs. [1, 6, 11, 14, 19, 20, 25, 26] and the references therein). Typical equations considered include those of the form

\[
y_n = \max\{g_1(y_{n-k_1}), g_2(y_{n-k_2}), \ldots, g_L(y_{n-k_L})\}, \quad n \geq 0
\]

(5)

for small \( L \geq 2 \), single-variable functions \( \{g_i\} \), and small delays \( \{k_i\} \), see for instance [11, 19, 20, 25].

Note that in the case

\[
K = \begin{bmatrix} k_1 & m_1 \\ k_2 & m_2 \end{bmatrix} = \begin{bmatrix} u & v \\ v & u \end{bmatrix},
\]

(6)

equation (4) corresponds (upon substituting \( x_n = -y_n \)) to the absolute-difference equation

\[
x_n = |x_{n-u} - x_{n-v}|, \quad n \geq 0,
\]

(7)

which was studied recently in the case of small \( u \) and \( v \) in Refs. [16, 17, 23]. Among the results in Ref. [23] is the following.

**Theorem 1.** Every integer solution to (7) with \( u = 2 \) and \( v = 1 \) is either eventually periodic with period three or else converges to zero.

Note that equation (7) corresponds to equation (4) with \( (k_1, m_1, k_2, m_2) = (2, 1, 1, 2) \). Among the implications of our results is the following generalization (see Theorem 6(i) and (ii), below).

**Theorem 2.** Suppose that \( \gcd(k_1, m_1, k_2, m_2) = 1 \) and \( k_1 = m_1 + k_2 \). Then, every integer solution to equation (4) is eventually periodic with (not necessarily prime) period \( m_1 + m_2 \) if and only if \( k_2 | m_2 \). Furthermore, there exists a non-trivial period \( m_1 + m_2 \) integer solution to the equation.

Homogeneous linear recurrences with constant coefficients, i.e. those of the form

\[
y_n = a_1y_{n-1} + a_2y_{n-2} + \ldots + a_s y_{n-s}
\]

(8)

with \( a_i \in \mathbb{Z}, \ i = 1, \ldots, s \), have attracted much attention over the years (cf. [9] for a voluminous bibliography). In many cases it can be shown that solutions are unbounded and in fact increase in size exponentially (see for instance Section 2.4 in Ref. [9]). As can be seen by comparing Theorems 5 and 6, below, which correspond to the cases \( L = 1 \) and \( L = 2 \), respectively, the behaviour of solutions can change drastically for differing values of \( L \) in (1).

The remainder of the paper proceeds as follows. Section 2 is devoted, for comparison purposes, to some consideration of the linear equations for the \( L = 1 \) case in (1). Section 3 focuses on an investigation of solutions to (4) under the useful criterion \( k_1 = m_1 + k_2 \). In Section 4, we consider results for the remaining cases, and the paper concludes with some remarks on possible generalizations and some open questions.
For some further work on piecewise-linear equations, the reader is referred to Refs. [2,5,8,10,12,13] as well as the vast interesting literature related to the well-known 3x + 1 Conjecture (see for instance [18,27] and the references therein).

2. The $L = 1$ case in equation (1)

In this section, we examine in some detail, for comparison purposes, the simple $L = 1$ case in equation (1), i.e. equations of the form

$$y_n = y_{n-k} - y_{n-m}, \quad n \geq 0,$$

(9)

where $k, m \geq 1$, and $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in \mathbb{R}$, with $s = \max\{k, m\}$.

It may be noted that a complete analysis of equation (9) would amount to exhaustive computation of the roots of the characteristic equation associated with (9) (see Theorem 3, below). While this is beyond the scope of this paper, we will provide a few more readily obtainable facts related to periodicities and growth rates for solutions.

The following standard result for linear homogeneous difference equations with real, constant coefficients will be useful (see for instance [7,22] or [9] for more details).

**Theorem 3.** Suppose that a recurrence relation of order $r$ with real coefficients has characteristic polynomial

$$P(x) = x^r + b_1x^{r-1} + \cdots + b_r,$$

(10)

where $b_r \neq 0$ and that the characteristic equation $P(x) = 0$ has real roots $\beta_1, \beta_2, \ldots, \beta_r$ of multiplicities $k_1, k_2, \ldots, k_r$, and complex conjugate root pairs $\rho_j(\cos \theta_j + i \sin \theta_j)$, $j = 1, 2, \ldots, w$ of multiplicities $s_j$, $j = 1, 2, \ldots, w$. Then, the general (real) solution of the corresponding recurrence relation

$$y_n = b_1y_{n-1} + b_2y_{n-2} + \cdots + b_ry_{n-r}$$

(11)

is given by

$$y_n = \sum_{j=1}^{s_1} \left( c_{j,1} + c_{j,2}n + \cdots + c_{j,k_j}n^{k_j-1} \right) \beta_j^n + \sum_{j=1}^{s_2} \left( d_{j,1} + d_{j,2}n + \cdots + d_{j,k_j}n^{k_j-1} \right) \rho_j^n \cos(n\theta_j) + \sum_{j=1}^{s_3} \left( g_{j,1} + g_{j,2}n + \cdots + g_{j,k_j}n^{k_j-1} \right) \rho_j^n \sin(n\theta_j), \quad n \geq 0$$

(12)

for real constants $c_{1,1}, \ldots, c_{r,k}, d_{1,1}, \ldots, d_{w,k}, g_{1,1}, \ldots, g_{w,k}$.

Theorem 3 follows from the general expression for $\{y_n\}$ in terms of characteristic roots by employing the fact that complex roots must occur in conjugate pairs and a change of basis (see for instance [7,15]).
Now, note that the characteristic equation for equation (9) is $P$ given by

\[ P(x) = \begin{cases} 
  x^k + x^{k-m} - 1, & \text{if } k > m \\
  x^m - x^{m-k} + 1, & \text{if } k < m 
\end{cases} \tag{13} \]

We have the following lemma regarding primitive roots of unity for polynomials of the form in (13).

**Lemma 1.** Suppose $a > b > 0$ and let

\[ Q_1(x) = x^a + x^b - 1 \quad \text{and} \quad Q_2(x) = x^a - x^b + 1. \tag{14} \]

Suppose $\xi_1$ is a zero of $Q_1$ and $\xi_2$ is a zero of $Q_2$.

(i) If $\xi_1$ or $\xi_2$ is a primitive $n$th root of unity, then $6|n$.

(ii) If $\xi_1$ is a primitive $(6j)$th root of unity for some $j \geq 1$, then the pair $(a, b)$ modulo $6j$ satisfies

\[ (a, b) \in \{(5j, j), (j, 5j)\} \mod 6j. \tag{15} \]

(iii) If $\xi_2$ is a primitive $(6j)$th root of unity for some $j \geq 1$, then

\[ (a, b) \in \{(2j, j), (4j, 5j)\} \mod 6j. \tag{16} \]

(iv) If $\xi_1$ or $\xi_2$ has modulus one then it cannot be a multiple root.

**Proof.** First note that since $\xi_1$ satisfies $Q_1(x) = 0$ we must have, by (14) that

\[ \Im(\xi_1^a) = -\Im(\xi_1^b) \quad \text{and} \quad \Re(\xi_1^a) + \Re(\xi_1^b) = 1, \]

where $\Im(z)$ and $\Re(z)$ are the imaginary and real parts of the complex number $z$, respectively. Hence,

\[ \text{either } (\xi_1^a, \xi_1^b) = (\xi, \xi^5) \quad \text{or} \quad (\xi_1^a, \xi_1^b) = (\xi^5, \xi), \tag{17} \]

where $\xi = 1/2 + i(\sqrt{3}/2)$ and $\xi^5$ are the two primitive 6th roots of unity. Similarly, $\xi_2$ satisfies

\[ \Im(\xi_2^a) = \Im(\xi_2^b) \quad \text{and} \quad \Re(\xi_2^a) - \Re(\xi_2^b) = -1, \]

and hence,

\[ \text{either } (\xi_2^a, \xi_2^b) = (\xi^2, \xi) \quad \text{or} \quad (\xi_2^a, \xi_2^b) = (\xi^4, \xi^5). \tag{18} \]

To prove (i), suppose $\xi_1$ is a primitive $n$th root of unity for some $n = 1$. The result then follows directly from (17) upon noting that the cyclic group

\[ S = \{\xi_1^a, \xi_1^{2a}, \ldots, \xi_1^n, 1\} = \{\xi, \xi^2, \ldots, \xi^5, 1\}, \tag{19} \]
generated by $\xi$ (or $\zeta^5$), is an order six subgroup of the cyclic group of order $n$

$$G = \{\xi_1, \xi_1^2, \ldots, \xi_1^{n-1}, 1\}. \quad (20)$$

generated by $\xi_1$. In the case for $\xi_2$, $\{\xi_2^1, \xi_2^2, \ldots, \xi_2^{n-1}, 1\} = \{\zeta, \zeta^2, \ldots, \zeta^5, 1\}$, and the result follows similarly.

To prove (ii), note that $S$ in (19) is the unique subgroup of order 6 of $G$, i.e. $S$ is the subgroup generated by $\xi_1^{n/6}$. Thus, by (17), $|a, b| = |n/6, 5n/6|$ modulo $n$, as required. Similarly, by (18), $\xi_2^n \in \{\zeta, \zeta^5\}$ and hence $b \in \{n/6, 5n/6\}$ modulo $n$. Since in each case in

$$(18), \ a = 2b \mod n, \text{ the result in (iii) follows.}$$

To prove (iv), consider the polynomial

$$f(x) = c_1x^a + c_2x^b + c_3, \quad (21)$$

where $c_1, c_2, c_3 \in \{-1, 1\}$. Then,

$$f'(x) = c_1ax^{a-1} + c_2bx^{b-1}. \quad (22)$$

Suppose that $\xi$ is a multiple root of $f$ of modulus one. Then, since $\xi$ is a zero of $f'$, employing (22), gives

$$a = |a\xi^{a-1}| = |b\xi^{b-1}| = b, \quad (23)$$

which contradicts the fact that $a > b > 0$.

Applying Lemma 1 and (13), we have the following.

**Theorem 4.** Suppose $k, m \geq 1$ and let $P$ be as in (13).

(i) Suppose $j \geq 1$. The characteristic polynomial $P$ has a zero which is a primitive $6j$th root of unity, if and only if

$$(k, m) \in \{\{j, 2j\}, \{5j, 4j\}\} \mod 6j. \quad (24)$$

(ii) Any primitive root of unity, $\rho$, which satisfies $P(\rho) = 0$ must be a primitive $6j$th roots of unity, for some $j \geq 1$. In addition, each root of $P$ of modulus one must have multiplicity one.

**Proof.** First, suppose $k > m$. Then $P(x) = x^k + x^{k-m} - 1$ and with $a = k$ and $b = k - m$, $(a,b)$ satisfies (15) if and only if $(k, k-m) \in \{(5j, j), (j, 5j)\} \mod 6j$, i.e. $(k, m) \in \{(5j, 4j), (j, 2j)\} \mod 6j$. Similarly, if $k < m$, setting $a = m$ and $b = m - k$, $(a,b)$ satisfies (16) if and only if $(m, m-k) \in \{(2j, j), (4j, 5j)\} \mod 6j$, i.e. $(m, k) \in \{(2j, j), (4j, 5j)\} \mod 6j$.

Now, for fixed $j \geq 1$, set $\rho = \exp((2\pi i)/6j)$. We then have that $\rho$ is a primitive $6j$th root of unity and $\rho' = \zeta = 1/2 + i(\sqrt{3}/2)$. Hence, if $k > m$ and $(k,m)$ satisfies (24), then

$$\rho^k + \rho^{k-m} = \zeta + \zeta^5 = 1 \text{ or } \rho^k + \rho^{k-m} = \zeta^5 + \zeta = 1, \quad (25)$$

and hence $P(\rho) = 0$. Similarly, if $k < m$ and $(k,m)$ satisfies (24), then

$$\rho^m - \rho^{m-k} = \zeta^2 - \zeta = -1 \text{ or } \rho^m + \rho^{m-k} = \zeta^4 - \zeta^5 = -1, \quad (26)$$

and $P(\rho) = 0$.
and again $P(\rho) = 0$. The result in (i) now follows upon applying Lemma 1. The first and second statements in (ii) follow from Lemma 1(i) and (iv), respectively.

We can now state the following result regarding periodicities and growth rates for solutions to (9).

**Theorem 5.** Suppose $k, m, j \geq 1$

(i) There exists a prime period $6j$ solution to (9), if and only if $(k, m)$ satisfies (24).

(ii) If $\gcd(k, m) = 1$ then all solutions to (9) are eventually periodic if and only if $(k, m) = (1, 2)$. If $(k, m) = (1, 2)$ then all non-trivial solutions must be strictly periodic with period six.

(iii) All solutions to (9) are either bounded or satisfy

$$\limsup_{n \to \infty} \frac{|y_n|}{n} = \infty$$

(27)

and, in particular, all unbounded solutions exhibit super-linear growth.

**Proof.** The result in (i) is a direct consequence of Theorem 4(i) and Theorem 3. For (ii), note that by Theorem 3 all solutions are periodic if and only if some roots of the characteristic polynomial $P$ are roots of unity and the remainder all have modulus less than one. Now, if $\gcd(k, m) = 1$ then according to (24), $j = 1$ and the only possible primitive roots of unity for the characteristic equation are the two primitive sixth roots of unity. Since from (13) the product of the moduli of the roots of $P$ (including multiplicity) must equal one, existence of a root having modulus less than one leads to some root of modulus greater than one. Thus, since Theorem 4(ii) precludes the possibility of multiple roots, we need only consider the case $(k, m) = (1, 2)$ where $P$ is of degree less than 3. The polynomial $P(x) = x^3 - x + 1$, indeed, has zeros $\zeta$ and $\zeta^3$ and hence all solutions have period six. The lack of multiple roots of modulus one implies, via Theorem 3, that unbounded solutions must exhibit exponential growth, and in particular (27) is satisfied, and the theorem is proven.

**Remark.** As examples of period six solutions when $(k, m) \equiv \{(1, 2), (5, 4)\} \mod 6$ we may take $\theta_1 = \pi/3$, $p_1 = 1$, $d_{1,1} = 1$ and all other coefficients zero in (12) to obtain the cycle $(1/2, -1/2, -1, 1/2, 1/2, 1)$. Similarly taking $s_{1,1} = 1$ and all other coefficients zero, we obtain the 6-cycle $(\sqrt{3}/2, \sqrt{3}/2, 0, -\sqrt{3}/2, -\sqrt{3}/2, 0)$. Integer solutions can be obtained by multiplying by appropriate constants.

If we allow for $\gcd(k, m) > 1$ we can obtain higher order prime periodic solutions. Taking for instance $(k, m) = (14, 16) = (2(1), 2(2)) \mod 2(6)$, we have the 12-cycle

$$\left(\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right).$$

(28)

We are now in a position to consider properties of solutions to (4).

3. **The $L = 2$ case in equation (1) under the criterion $k_1 = m_1 + k_2$**

In this section we will consider in some depth the behaviour of integer solutions to equation (4) under the criterion $k_1 = m_1 + k_2$. The ensuing results which include Theorem
Table 1. Behaviour of integer solutions to equation (4) for $k_1 = m_1 + k_2$ (1000 iterations; number of particular (eventual) periods are listed in parentheses).

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$m_1$</th>
<th>$k_2$</th>
<th>$m_2$</th>
<th>$\gcd(k_2, m_1 + m_2)$</th>
<th>Unbounded solutions</th>
<th>Prime periods</th>
</tr>
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<td>3</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>3(2), 6(991)</td>
</tr>
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<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>16</td>
<td>3(994)</td>
</tr>
<tr>
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<td>4</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>5(1), 10(999)</td>
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<tr>
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<td>2</td>
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<td>1</td>
<td>0</td>
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</tr>
<tr>
<td>8</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>9(1000)</td>
</tr>
<tr>
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<td>3</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>36</td>
<td>5(964)</td>
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<td>1</td>
<td>0</td>
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<td>7</td>
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<td>1</td>
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<td>9</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>13(1000)</td>
</tr>
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</table>

2 (as an application of Theorem 6, below), will be employed in Section 4 in considering some more general cases of equation (4).

Table 1 contains some preliminary data on behaviour of solutions to (4) for 1000 iterations and random integer initial values in the interval $[-10, 10].^2$

Each equation considered in Table 1, satisfies the assumption

$$k_1 = m_1 + k_2.$$

Note that in many cases solutions with eventual periods $k_2$ and $m_1 + m_2$ exist and in addition that there appears to be a relationship between existence of unbounded solutions and the greatest common divisor of these two quantities. Over the remainder of this section, we will prove the following theorem.

**Theorem 6.** Suppose $k_1 = m_1 + k_2$ and $\gcd(k_1, m_1, k_2, m_2) = 1$.

(i) There exists a non-trivial integer solution to equation (4) which has period $p = m_1 + m_2$.

(ii) All integer solutions to equation (4) are eventually periodic with period $p = m_1 + m_2$ if and only if $k_2 | m_2$.

(iii) If $k_2 \neq m_2$, then there exists a non-trivial period $k_2$ integer solution to equation (4).

(iv) If $\gcd(k_2, m_1 + m_2) > 1$, then there exists an unbounded integer solution to equation (4) which satisfies

$$\limsup_{n \to \infty} \frac{|y_n|}{n} = \frac{1}{\text{lcm}(k_2, m_1 + m_2)}.$$  

(30)
(v) Any unbounded integer solution to (4), must satisfy

\[
\limsup_{n \to \infty} \frac{|y_n|}{n} < \infty.
\]  

(31)

Compare Theorem 6 with Theorem 5, above. In particular note the stark contrast between the results in Theorem 6(iv) and (v) and that in Theorem 5(iv).

Proof of Theorem 6. The theorem is a compilation of the results in Theorems 7–12, below. \(\square\)

In the next subsection, we provide some preliminaries regarding solutions to (4). In Section 3.2, we prove Theorems 7–10 (Theorem 6(i)–(iii)), while Section 3.3 includes proofs of Theorems 11 and 12 (Theorem 6(iv) and (v)).

3.1 Preliminary lemmas and notation

Throughout this subsection we shall assume that \(k_1 = m_1 + k_2\). We first introduce some helpful notation. For an integer sequence \(\{y_i\}\) satisfying equation (4), define the partition \([S_0, S_1, S_2]\) of \(\mathbb{N}^+ = \{0, 1, 2, \ldots\}\) via

\[
S_1 = \{i : y_{i-k_1} - y_{i-m_1} < y_{i-k_2} - y_{i-m_2}\}
\]

\[
S_2 = \{i : y_{i-k_1} - y_{i-m_1} > y_{i-k_2} - y_{i-m_2}\}
\]

\[
S_0 = \{i : y_{i-k_1} - y_{i-m_1} = y_{i-k_2} - y_{i-m_2}\}.
\]  

(32)

Now, set \(p = m_1 + m_2\). We have the following lemmas whose proofs emphasize the role of (29) as a property of interest.

Lemma 2. For \(n > m_1\) if \(n \in S_1 \cup S_0\), then \(y_n - y_{n-p} \geq 0\). In particular, if \(y_n - y_{n-p} < 0\) then \(n \in S_2\).

Proof. We have

\[
y_n - y_{n-p} = y_{n-k_1} - y_{n-m_1} - y_{n-p} \geq y_{n-k_1} - (y_{n-m_1 - k_2} - y_{n-m_1 - m_2}) - y_{n-p} = 0,
\]  

(33)

where the inequality follows from (4) and the final equality is a result of (29) and the definition of \(p\). \(\square\)

Lemma 3. If \(n \in S_2 \cup S_0\) for some \(n > m_2\), then

\[
y_n - y_{n-p} \geq y_{n-k_1} - y_{n-k_2 - p},
\]  

(34)

with strict inequality if \(n - m_2 \in S_2\).
Proof. Note that

\[ y_n - y_{n-p} = y_{n-k_2} - y_{n-m_2} - y_{n-p} = y_{n-k_2} - (y_{n-m_2-k_1} - y_{n-p}) - y_{n-p} = y_{n-k_2} - y_{n-k_2-p}, \]

with strict inequality when \( n - m_2 \in S_2 \).

Combining Lemmas 2 and 3, we have the following.

**Lemma 4.** If \( y_n - y_{n-p} < 0 \) for some \( n > s \), then \( n - k_2 \in S_2 \) and

\[ y_{n-k_2} - y_{n-k_2-p} \leq y_n - y_{n-p} < 0 \]

with strict inequality in (36) if \( n - m_2 \in S_2 \).

Proof. Since \( y_n - y_{n-p} < 0 \), by Lemma 2, \( n \in S_2 \) and hence the first inequality in (36) follows from Lemma 3. The fact that \( n - k_2 \in S_2 \) follows by (36) and Lemma 2 upon noting that \( n - k_2 > s - k_2 = k_1 - k_2 = m_2 \).

The next three lemmas are related to the growth, boundedness and sign structure of solutions to (4).

**Lemma 5.** Suppose that \( \{y_i\} \) is an integer solution to (4). Then, the sets

\[ T_2 = \{ i : y_i \geq 0 \} \quad \text{and} \quad T_1 = \{ i : y_i \leq 0 \} \]

must be infinite.

Proof. Suppose \( T_1 \) is finite. Then, there exists an \( N > 0 \) such that \( y_i > 0 \) for \( i > N \). Referring to equation (4), we have \( y_i < y_{i-k_1} \) for \( i > N + m_1 \). Thus the integer sequence \( \{y_{N+m_1+t}\}_{t \geq 0} \) is strictly monotonically decreasing, which contradicts the assumption that \( T_1 \) is finite. Now, assume that \( T_2 \) is finite and hence that there exists an \( N_2 > 0 \) such that \( y_i < 0 \) for \( i > N_2 \). We have, by equation (4), that

\[ y_i > \min\{y_{i-k_1}, y_{i-k_1}\}. \]

For \( t \geq 1 \), set

\[ m_t = \min\{y_t : N_2 + tk_1 \leq i < N_2 + (t+1)k_1\} \]

and note that by (38), \( \{m_t\}_{t \geq 1} \) is a strictly increasing integer sequence. This contradicts the fact that \( T_2 \) is assumed to be finite.

The following lemma will be pertinent to the discussion regarding boundedness in Section 3.3.

**Lemma 6.** Suppose \( \gcd(k_2, p) = 1 \). If \( H = \{ i : y_i - y_{i-p} < 0 \} \) is finite then \( \{y_i\} \) is eventually periodic with period \( p \).
Proof. If \( y_n - y_{n-p} = 0 \) for sufficiently large \( n \), then \( \{ y_i \} \) is eventually periodic with period \( p \). Hence assume that \( \{ i : y_i - y_{i-p} > 0 \} \) is infinite. Then, since \( H \) is finite, \( \{ y_i \} \) is bounded from below and

\[
\lim_{i \to \infty} y_{u+rp} = \infty
\]

for some \( 0 \leq u \leq p - 1 \). Note that, by equation (4),

\[
y_{u+rp} \leq y_{u+rp-k_2} - y_{u+rp-m_2}.
\]

Since \( \{ y_i \} \) is bounded from below, we have, in particular, that \( \{ y_{u+rp-m_2} \}_{r > 0} \) is bounded from below. Hence combining (40) and (41) gives \( \lim_{i \to \infty} y_{u+rp-k_2} = \infty \). Proceeding in this fashion, since \( \gcd(k_2, p) \) we have that

\[
\lim_{i \to \infty} y_{v+rp} = \infty \text{ for } v \in \{0, 1, \ldots, p - 1\}
\]

and hence \( T_1 = \{ i : y_i \leq 0 \} \) is finite. This contradicts the fact that, by Lemma 5, \( T_1 \) must be infinite and the lemma is proven. \( \square \)

Note that the proofs of Lemmas 5 and 6 did not require the assumption in (29).

Finally, we have the following regarding descendance of the sequence \( \{ y_i \} \).

**Lemma 7.** There exists a finite \( Q < 0 \) such that \( y_n - y_{n-p} > Q \) for all \( n \geq -s + p \).

**Proof.** This follows from Lemma 4. In particular, set

\[
\mathcal{V} = \{ -s + p, -s + p + 1, \ldots, s - 1, s \}
\]

and

\[
Q = \min \left\{ -1, \min_{i \in \mathcal{V}} \{ y_i - y_{i-p} \} \right\}.
\]

Now, suppose that \( N > s \) and \( y_N - y_{N-p} < 0 \), where \( N = k_2u + v \) with \( u \geq 1 \) and \( v \in \mathcal{V} \). Note that by (29) and (89),

\[
\| \mathcal{V} \| = 2s - p + 1 > m_2 + k_1 - (m_1 + m_2) = k_2,
\]

and hence such an expression for \( N \) is possible. Inductively applying Lemma 4, we have that

\[
y_N - y_{N-p} \geq y_{N-k_2} - y_{N-k_2-p} \geq \cdots \geq y_v - y_{v-p} \geq Q
\]

as required. \( \square \)

We now move on to considering our main results in the case that (29) is satisfied.

### 3.2 Main results – periodic solutions

In this subsection we will prove results concerning possible periodic solutions to (4) under the criterion \( k_1 = k_2 + m_1 \); other cases are considered in Section 4.
3.2.1 Period $p = m_1 + m_2$ solutions

Our first theorem regards the case $k_2|m_2$. Assuming (29), it will turn out that this is the only subcase for which all integer solutions have eventual period $p$ (see Theorem 10, below).

**Theorem 7.** Suppose that $k_2|m_2$ and $k_1 = m_1 + k_2$. Then all integer solutions to equation (4) must have period $m_1 + m_2$.

**Proof.** The assumption $k_2|m_2$ is essential here. From Lemma 7, we can find a finite $Q < 0$ such that

$$Q \leq \min_i \{ y_i - y_{i-p} \}. \tag{47}$$

Now, suppose that

$$y_N - y_{N-p} < 0 \tag{48}$$

for some $N > |Q|k_2 + p$. Then, by Lemma 4,

$$y_{N-k_2} - y_{N-k_2-p} \leq y_N - y_{N-p} < 0 \tag{49}$$

and hence $N - k_2 \in S_2$ by (2). In fact,

$$y_{N-rk_2} - y_{N-rk_2-p} \leq y_{N-(r-1)k_2} - y_{N-(r-1)k_2-p} < 0 \tag{50}$$

for $N - rk_2 > m_2$, and in a similar fashion, we have

$$N - rk_2 \in S_2, \tag{51}$$

for $N - rk_2 > p > \max\{m_1, m_2\}$, i.e. $r < (N - p)/k_2$. Then, since $m_2 = rk_2$ for some $r \geq 1$,

$$N - rk_2 - m_2 = N - (r + t)k_2 \in S_2 \tag{52}$$

whenever $N - (r + t)k_2 > p$. Thus employing Lemma 4, the inequality in (50) is actually strict. But then

$$Q < y_{N-|Q|k_2} - y_{N-|Q|k_2-p} < y_{N-(|Q|-1)k_2} - y_{N-(|Q|-1)k_2-p} < \cdots$$

$$< y_{N-k_2} - y_{N-k_2-p} < y_N - y_{N-p} < 0. \tag{53}$$

Equation (53) gives a monotone sequence of $|Q| + 1$ integers in the interval $(Q,0)$, which is an impossibility, and the theorem is proven.

The next theorem proves existence of non-trivial period $p = m_1 + m_2$ solutions in all cases.

**Theorem 8.** Suppose that $k_1 = m_1 + k_2$. Then there exists a non-trivial integer solution to equation (4) which has period $p = m_1 + m_2$. 

\textbf{Proof.} Consider the solution with initial values \( y_{-i} = -1 \) for \(-s \leq i \leq -1\), where \( s = \max \{ k_1, m_2 \} \). We will first show that the set of values \( \mathcal{S} = \{ n : -s + p \leq n \leq -s + p + (k_2 - 1) \} \), is a set of \( k_2 \) consecutive values for which

\[ y_n \geq y_{n-p}. \]  

(54)

By Lemma 4, this will imply that \( y_n \geq y_{n-p} \) for all \( n > -s + p \). To that end note that for \( n \in \mathcal{S} \), either \( n \leq -1 \) in which case \( y_n = -1 \) or

\[ -s \leq -k_1 \leq n - k_1 \leq -s + p + k_2 - 1 - k_1 = -s + m_2 - 1 \leq -1 \]  

(55)

and hence \( y_{n-k_1} = -1 \). Inductively, we then have for \( n \in \mathcal{S} \) that \( y_n \in \{-1, 0\} \). In addition, for \( n \in \mathcal{S} \),

\[ -s \leq n - p \leq -s + k_2 - 1 \leq -1 \]  

(56)

and hence \( y_{n-p} = -1 \), which proves (54).

Now, suppose that \( y_n \geq 2 \) for some \( n \) and let \( N = \min \{ i \geq -s : y_i \geq 2 \} \). Then, \( y_i \in \{-1, 0, 1\} \) for all \( i < N \) and \( y_{N-k_1} = y_{N-k_2} = 1 \) by (4) and the definition of \( N \). We then have

\[ 1 = y_{N-k_1} \leq y_{N-k_2-k_1} = y_{N-k_2-k_1} - y_{N-k_2-k_1} = y_{N-k_2-k_1} - 1 \leq 0, \]  

(57)

which is a contradiction. Hence we have \( y_i \geq y_{i-p} \) for all \( i \geq -s + p \) but at the same time \( y_i \leq 1 \) for all \( i \geq -s \), hence \( y_i - y_{i-p} = 0 \) for sufficiently large \( i \) and the existence of a period \( p \) solution follows. To see that this construction results in a periodic solution that is non-trivial, note that since \( y_i \geq y_{i-p} \) for all \( i \geq -s + p \) and \( y_i = -1 \) for \(-s \leq i \leq -1 \), we have

\[ N^* = \sup \{ i \geq -s : y_i \leq -1 \} \geq -1. \]  

(58)

Suppose \( N^* \) is finite. Then \( y_i \leq 0 \) for \( i > N^* \) and hence, employing equation (4),

\[ y_{N^*+k_1} \leq y_{N^*} - y_{N^*+k_1-k_1} = y_{N^*} - y_{N^*+k_2} \leq -1, \]  

(59)

which contradicts the definition of \( N^* \). Thus \( \{ i : y_i \leq -1 \} \) is infinite and the constructed periodic solution is indeed non-trivial. \( \square \)

3.2.2 \textbf{Period} \( k_2 \) \textbf{solutions}

Next, we prove the existence of prime period \( k_2 \) solutions when \( k_2 \nmid m_2 \).

\textbf{THEOREM 9.} Suppose that \( k_1 = m_1 + m_2 \) and \( k_2 \nmid m_2 \). Then there exists a prime period \( k_2 \) integer solution to equation (4).

\textbf{Proof.} Suppose we have initial values satisfying

\[ y_0 = \begin{cases} -1, & \text{if } n = s \ \text{mod} \ k_2 \\ 0, & \text{otherwise} \end{cases} \]  

(60)
for \(-s \leq n \leq -1\). We will prove that \(\{y_n\}_{n \geq -s}\) satisfies (60) for all \(n\). Suppose this is true for \(n < N\). Then, since \(N - k_1 = N - m_1 \mod k_2\), \(y_{N-k_1} - y_{N-m_1} = 0\). Now, suppose \(s = k_1\). If \(N \equiv k_1 \mod k_2\), then \(y_{N-k_2} = -1\) and \(N - m_2 \equiv k_1 - m_2 \equiv k_1 \mod k_2\) since \(k_2 \neq m_2\) and hence \(y_{N-m_2} = 0\). Thus \(y_N = -1\) as required. If \(N \neq k_1 \mod k_2\) then \(y_{N-k_2} = 0\) and \(y_{N-m_2} = 0\) and hence \(y_N = 0\) as required. The case \(s = m_2\) follows by a similar argument and the theorem is proven. \(\square\)

We now prove Theorem 6(ii) (see also Theorem 2).

**Theorem 10.** Assume that \(k_1 = m_1 + k_2\). Then, all integer solutions to (4) are eventually periodic with period \(p = m_1 + m_2\) if and only if \(k_1|m_2\).

**Proof.** Assume that \(k_2 \nmid m_2\). Then, in particular \(k_2 \neq 1\). If \(k_2 \nmid p\) then, by Theorem 9, there exists a periodic solution with prime period not dividing \(p\). If \(k_2|p\) then \(\gcd(k_2, p) = k_2 > 1\) and there exists an unbounded solution, by Theorem 12, below. The result then follows upon applying Theorem 7. \(\square\)

### 3.3 Main results – linear growth and unbounded solutions

We begin with the following seemingly well-founded conjecture (see Table 1, above).

**Conjecture 1.** Suppose \(k_1 = m_1 + k_2\), then there exists an unbounded integer solution to (4) if and only if \(\gcd(k_2, m_1 + m_2) > 1\). Any unbounded integer solutions must satisfy (31), i.e.

\[
\limsup_{n \to \infty} \frac{|y_n|}{n} < \infty.
\]

We will first prove the second part of the conjecture.

**Theorem 11.** Suppose \(k_1 = m_1 + k_2\). Then, any unbounded integer solution to equation (4) must satisfy (61).

**Proof.** First, suppose \(\{y_i\}\) is a solution to (4), and set

\[
D = \min\{ -1, y_0, y_1, \ldots, y_{p-1}\} \leq -1,
\]

and define \(Q\) as in Lemma 7 so that for all \(n \geq -s + p\)

\[
y_n - y_{n-p} \geq Q.
\]

Then, for all \(i \geq \max\{p, 2|D|/|Q|\}\),

\[
\frac{y_i}{i} \geq \frac{D + Q|i/p|}{i} \geq \frac{D + Q(i/2)}{i} \geq \frac{D}{i} + \frac{Q}{2} \geq Q,
\]
where we have used the fact that \( p \geq 2 \). Thus, we have from (64) that
\[
\liminf_{n \to \infty} \frac{y_n}{n} < \infty. \tag{65}
\]

Now, suppose that, in addition, \( \{y_i\} \) satisfies
\[
\limsup_{n \to \infty} \frac{y_n}{n} = \infty, \tag{66}
\]

Employing (66), there exists an \( M \geq \max \{p, 2\|Q\|, 2\|Q\|\} \) and \( R > 2\|Q\| \), such that
\[
\frac{y_M}{M} > R \quad \text{and} \quad \frac{y_j}{j} \leq R, \quad j = -s, \ldots, M - 1. \tag{67}
\]

Thus, we obtain
\[
MR < y_M = \min \{y_{M-k_1} - y_{M-m_1}, y_{M-k_2} - y_{M-m_2}\},
\]

\[
\leq \min \{y_{M-k_1}, y_{M-k_2}\} + |Q|M, \tag{68}
\]

and hence
\[
y_{M-k_1} > MR - M|Q| \quad \text{and} \quad y_{M-k_2} > MR - M|Q| > M|Q|. \tag{69}
\]

On the other hand
\[
y_{M-k_2} \leq y_{M-k_2-k_1} - y_{M-k_2-m_1} \leq R(M - k_1 - k_2) - y_{M-k_1}
\]

\[
\leq RM - RM + M|Q| = M|Q|, \tag{70}
\]

which contradicts the second statement in (69), and the theorem is proven. \( \square \)

It seems reasonable that a proof of the conjecture that \( \gcd(k_2, m_1 + m_2) = 1 \) implies boundedness might employ something akin to Lemma 6.

The remainder of this subsection consists of a result regarding the existence of unbounded solutions to (4). By necessity, the proof is somewhat technical and number-theoretic. The casual reader can safely skip to Section 4 as the subsequent results do not depend on the current subsection.

We will prove the following existence result.

**Theorem 12.** Suppose that \( k_1 = m_1 + k_2 \) and \( d = \gcd(k_2, m_1 + m_2) > 1 \). Then, there exists an unbounded solution to (4). Furthermore, the constructed integer solution satisfies
\[
\limsup_{n \to \infty} \left| \frac{y_n}{n} \right| = \frac{1}{\operatorname{lcm}(k_2, m_1 + m_2)}. \tag{71}
\]

**Proof.** Suppose that the conditions in the statement of the theorem hold. Note that
\[
\gcd(m_1, d) = 1 \text{ and } \gcd(m_2, d) = 1 \tag{72}
\]

since otherwise \( \gcd(k_1, k_2, m_1, m_2) > 1 \) which is assumed not to hold.

In establishing existence of unbounded solutions, we will consider the cases \( d > 3 \), \( d = 3 \) and \( d = 2 \) separately.
(The case: d > 3) First, suppose d > 3 and consider initial values (for convenience) \(y_0, y_1, \ldots, y_{k_2-1}\) satisfying \(y_i = -2\) if \(i = 0 \mod d\) and \(i < k_2\) and otherwise, for \(n \geq 1\)

\[
y_n = \begin{cases} 
1, & \text{if } n = m_1 \mod p \\
-1, & \text{if } n = m_1 - m_2 \mod d \\
y_{n-k_2} - 1, & \text{if } n = 0 \mod d \text{ and } n \neq 0 \mod p \\
y_{n-k_2}, & \text{otherwise}
\end{cases}
\]  

(73)

For example, in the case \((k_1, m_1, k_2, m_2) = (5, 1, 4, 7)\), we have \(d = \gcd(k_2, m_1 + m_2) = 4\), and employing (73), gives the unbounded sequence

\[
\{y_i\}_{i=0} = \{-2, 1, -1, 0, -2, 0, -1, 0, -3, 1, -1, 0, -3, 0, -1, 0, -4, 1, -1, 0, -4, \ldots\}.
\]  

(74)

Now, employing (72) and the fact that \(d > 3\), we have that \(m_1, m_1 - m_2 = 2m_1, 3m_1\) and zero are all distinct modulo \(d\).

We will show that (73) holds for all \(n \geq 0\). The result will then follow since (as will be shown at the end of the proof), \(\{y_i; i = 0 \mod d\}\) will be unbounded from below.

Suppose that \(y_n\) satisfies (73) for all \(n < N\). We will consider six cases.

(i) \((N = m_1 \mod p)\). In this case, \(N - m_1 = 0 \mod p\), \(N - k_2 = m_1 - k_2 = m_1 \mod d\) and \(N - m_2 = m_1 - m_2 \mod p\). From the induction hypothesis and (29), we then have

\[
y_{N-k_2} - y_{N-m_1} = y_{N-k_1} - (y_{N-m_1 - k_1} - 1) = 1.
\]

In addition, \(y_{N-k_2} \in \{0, 1\}\) and \(y_{N-m_2} = -1\). Thus, \(y_N = 1\), as required.

(ii) \((N \neq m_1 \mod p \text{ and } N = m_2 \mod d)\). In this case,

\[
y_{N-k_2} - y_{N-m_1} = y_{N-k_1} - y_{N-m_1 - k_2} = 0,
\]

and as in Case (i), \(y_{N-k_2} \in \{0, 1\}\) and \(y_{N-m_2} = -1\). Thus, \(y_N = 0\), as required.

(iii) \((N = m_1 - m_2 \mod d)\). In this case, \(N - k_1 = m_1 - k_2 = (m_1 + k_2) - m_2 = m_1 \mod d\) and \(N - m_1 = -m_2 = m_1 \mod d\). Hence \(y_{N-k_1}, y_{N-m_1} \in \{0, 1\}\) and thus

\[
y_{N-k_2} - y_{N-m_1} \in \{-1, 0, 1\}.
\]

In addition, \(N - k_2 = m_1 - m_2 \mod d\) and \(N - m_2 = m_1 - 2m_2 = 3m_1 \mod d\). We then have, \(y_{N-k_2} = -1\) and since \(3m_1 \not\in \{0, m_1, m_1 - m_2\} \mod d\) by the induction hypothesis, \(y_{N-m_2} = 0\). Thus, \(y_N = -1\), as required.

(iv) \((N \equiv 0 \mod p)\). In this case, \(N - k_1 = k_1 = -m_1 \mod d\) and \(N - m_1 = -m_1 \mod p\), and in particular \(N - k_1 = N - m_1\) in the same non-zero class modulo \(d\). Hence, by the induction hypothesis, \(y_{N-k_1} - y_{N-m_1} \in \{-1, 0, 1\}\). In addition,

\[
y_{N-k_2} - y_{N-m_2} = y_{N-k_2} - 1 \not< -2.\]

Thus, \(s = y_{N-k_2} - 1\), as required.

(v) \((N \neq 0 \mod p \text{ and } N \equiv 0 \mod d)\). As in Case (iv), \(N - k_1 = N - m_1 = -m_1 \mod d\) and \(y_{N-k_1} - y_{N-m_1} \in \{-1, 0, 1\}\). On the other hand, now, \(N - m_2 = -m_2 \mod d\)
while \( N - m_2 \equiv - m_2 \mod p \). From the induction hypothesis, we then have
\[
y_{N-k_2} - y_{N-m_2} = y_{N-k_2} - 0 \leq -2.
\]

Thus, \( y_N = y_{N-k_2} \), as required.

(vi) \((N \text{ not in the set } \{0, m_1, m_1 - m_2\} \text{ modulo } d)\). Since \( N - k_2 \notin \{0, m_1, m_1 - m_2\} \mod d \), we have, by the induction hypothesis, that \( y_{N-k_2} = 0 \). In addition, \( N - m_2 \equiv - m_2 \equiv m_1 \mod p \) and hence \( y_{N-m_2} \leq 0 \). Since \( N - k_1 \equiv N - m_1 \mod d \) and \( N - m_1 \notin \{0, m_1\} \),
\[
y_{N-k_1} - y_{N-m_1} = 0.
\]

Thus, \( y_N = 0 \), as required.

(The case: \( d = 2 \)) Now, for \( d = 2 \), consider initial values \( y_0, y_1, \ldots, y_{r-1} \) satisfying
\[
y_i = -2 \text{ if } i \equiv 0 \mod d \text{ and } i < k_2 \text{ and otherwise, for } n \geq 1
\]

\[
y_n = \begin{cases} 
1, & \text{if } n \equiv m_1 \mod p, \\
y_{n-k_2} - 1, & \text{if } n \equiv 0 \mod p, \\
y_{n-k_2}, & \text{if } n \equiv 0 \mod d \text{ and } n \not\equiv 0 \mod p, \\
0, & \text{otherwise}
\end{cases} \tag{75}
\]

By (72), \( m_1 \not\equiv 0 \mod d \). We will show that (75) holds for all \( n \geq 0 \). The result will then follow since again \( \{y_i : i \equiv 0 \mod d\} \) will be unbounded from below.

Suppose that \( y_n \) satisfies (75) for all \( n < N \). We will consider four cases.

(i) \((N \equiv m_1 \mod p)\). In this case, \( N - m_1 \equiv 0 \mod p, N - k_2 \equiv m_1 - k_2 \equiv m_1 \mod d \) and \( N - m_2 = m_1 - m_2 = 2m_1 \equiv 0 \mod d \). From the induction hypothesis and (29), we then have
\[
y_{N-k_1} - y_{N-m_1} = y_{N-k_1} - (y_{N-m_1} - 1) = 1.
\]

In addition, \( y_{N-k_2} \in \{0, 1\} \) and \( y_{N-m_2} \leq -2 \). Thus, \( y_N = 1 \), as required.

(ii) \((N \equiv m_1 \mod p \text{ and } N \equiv m_1 \mod d)\). In this case,
\[
y_{N-k_1} - y_{N-m_1} = y_{N-k_1} - y_{N-m_1} = 0,
\]

and as in Case (i), \( y_{N-k_2} \in \{0, 1\} \) and \( y_{N-m_2} \leq -2 \). Thus, \( y_N = 0 \), as required.

(iii) \((N \not\equiv 0 \mod p)\). In this case, \( N - k_1 \equiv - k_1 \equiv - m_1 \mod d \) and \( N - m_1 \equiv - m_1 \mod p \), and in particular \( N - k_1 \) and \( N - m_1 \) are in the same non-zero class modulo \( d \). Hence, by the induction hypothesis, \( y_{N-k_1} - y_{N-m_1} \in \{-1, 0, 1\} \). In addition, \( N - m_2 \equiv - m_2 \equiv m_1 \mod p \), and thus \( y_{N-k_2} - y_{N-m_2} = y_{N-k_2} - 1 \leq -2 \). Thus, \( y_N = y_{N-k_2} - 1 \), as required.

(iv) \((N \not\equiv 0 \mod p \text{ and } N \equiv 0 \mod d)\). As in Case (iv), \( N - k_1 \equiv N - m_1 \equiv - m_1 \mod d \) and \( y_{N-k_1} - y_{N-m_1} \in \{-1, 0, 1\} \). On the other hand, now, \( N - m_2 = m_1 \mod d \) while \( N - m_2 \not\equiv m_1 \mod p \). From the induction hypothesis, we then have
\[
y_{N-k_2} - y_{N-m_2} = y_{N-k_2} - 0 \leq -2.
\]

Thus, \( y_N = y_{N-k_2} \), as required.
Since $N$ is either zero or one modulo $d = 2$, the result is proved in this case.

(The case: $d = 3$) For $d = 3$, we will need to consider the two subcases

$$k_2 \neq 0 \mod p \quad \text{and} \quad k_2 \equiv 0 \mod p \quad (76)$$

First suppose $k_2 \neq 0 \mod p$, and consider initial values $y_0, y_1, \ldots, y_{k_2 - 1}$ satisfying $y_i = -2$ if $i \equiv 0 \mod d$ and $i < k_2$ and otherwise, for $n \geq 1$

$$y_n = \begin{cases} 
1, & \text{if } n \equiv m_1 \mod p \text{ or } n \equiv k_1 - m_2 \mod p \\
-1, & \text{if } n \equiv m_1 - m_2 \mod p \\
y_{n - k_2} - 1, & \text{if } n \equiv 0 \mod p \\
y_{n - k_2}, & \text{if } n \equiv 0 \mod d \text{ and } n \not\equiv 0 \mod p \\
0, & \text{otherwise} \end{cases} \quad (77)$$

Note that $k_1 - m_2 \equiv m_1 - m_2 \mod d$, but $k_1 - m_2 - (m_1 - m_2) = k_1 - m_1 \equiv k_2 \neq 0 \mod p$. In addition, by (72), $m_1 - (m_1 - m_2) = m_2 \not\equiv 0 \mod d, m_1 \not\equiv 0 \mod d$ and $m_1 - m_2 = 2m_1 = 0 \mod d$.

We will show that (77) holds for all $n \geq 0$. The result will then follow since \{\{y_i : i \equiv 0 \mod d\}\} will be unbounded from below.

Suppose that $y_n$ satisfies (77) for all $n < N$. We will consider seven cases.

(i) $(N \equiv m_1 \mod p)$. This case follows verbatim as in the case $d > 3$.

(ii) $(N \not\equiv m_1 \mod p$ and $N \equiv m_1 \mod d$). In this case,

$$y_{N - k_1} - y_{N - m_1} = y_{N - k_1} - y_{N - m_1} = 0.$$

If $N \equiv k_1 \mod p$, then $N - k_2 \equiv k_1 - k_2 = m_1 \mod p$ and $N - m_2 \equiv k_1 - m_2 \mod p$. Thus, by the induction hypothesis, $y_{N - k_2} - y_{N - m_2} = 1 - 1 = 0$ and hence $y_N = 0$, as required. Otherwise, $N - k_2 \equiv k_1 - k_2 = m_1 \mod p$, while $N - k_2 = m_1 \mod d$ and hence $y_{N - k_2} = 0$, and $N - m_2 \equiv k_1 - m_2 \mod p$ and $N - m_2 \equiv m_1 - m_2 \mod d$ while $N - m_2 = m_1 - m_2 \mod d$ and hence $y_{N - m_2} = 0$. Thus, in each instance $y_N = 0$ as required.

(iii) $(N \equiv k_1 - m_2 \mod p)$. In this case, $N - k_1 \equiv -m_2 = m_1 \mod p$ and $N - m_1 \equiv k_1 = m_1 + k_2 = m_1 \mod d$, but $N - m_1 = k_1 = m_1 + k_2 \equiv m_1 \mod p$. Thus, $y_{N - k_1} - y_{N - m_1} = 1 - 0 = 1$. In addition, $N - k_2 \equiv k_1 - m_2 = m_1 - m_2 \mod p$ and $N - m_2 \equiv k_1 - m_2 = m_1 - m_2 \mod d$ (since $d = 3$). Thus, $y_{N - k_2} - y_{N - m_2} = -1 - y_{N - m_2} \geq 1$, and $y_N = 0$, as required.

(iv) $(N \equiv m_1 - m_2 \mod p)$. In this case, $N - k_1 = m_1 - m_2)$, $N - k_1 = m_1 - m_2) = m_1 - m_2 \mod p$. Hence, $N - k_1 = m_1 \mod d$, but $N - k_1 \equiv m_1 \mod p$ and thus, $y_{N - k_1} = 0$. On the other hand, $N - m_1 = m_1 \mod p$ and hence $y_{N - m_1} = 0 - 1 = -1$.

In addition, $N - k_2 = m_1 - m_2 \mod d$ and hence $y_{N - k_2} \in \{-1, 1, 0\}$, while $N - m_2 = m_1 - 2m_2 = m_1 = 0 \mod d$. We then have, $y_{N - k_2} - y_{N - m_2} \geq -1 - (-2) = 1$ and thus, $y_N = -1$, as required.

(v) As in the preceding case, $y_{N - k_2} - y_{N - m_2} \geq -1 - (-2) = 1$. Meanwhile, $N - k_1 \equiv -m_2 = m_1 \mod p$, while $N - k_1 = m_1 \mod d$ and hence $y_{N - k_1} = 0$. In addition, $N - m_1 \equiv -m_2 = m_1 \mod p$, while $N - m_1 = m_1 \mod d$ and hence $y_{N - m_1} = 0$. Thus, $y_N = 0$, as required.
(vi) \( (N = 0 \mod p) \). This case follows verbatim as in the case \( d > 3 \).

(vii) \( (N \notin 0 \mod p \text{ and } N = 0 \mod d) \). We have that \( N - k_1 \) and \( N - m_1 \) are in the same non-zero class modulo \( d \) and hence \( y_{N-k_1} - y_{N-m_1} \geq -2 \). On the other hand, \( N - m_2 = m_1 \mod d \) while \( N - m_2 \neq m_1 \mod p \). From the induction hypothesis, we then have

\[
y_{N-k_1} - y_{N-m_2} = y_{N-k_2} = 0 \leq -2.
\]

Thus, \( y_N = y_{N-k_2} \), as required.

Since \( N \) is either \( m_1, m_1 - m_2 \) or zero modulo \( d = 3 \), the result is proved in the case \( k_2 \neq 0 \mod p \).

Suppose \( p | k_2 \). Then

\[
p = d = 3.
\]

Now, consider initial values \( y_0, y_1, \ldots, y_{s-1} \) satisfying \( y_i = -2 \) if \( i \equiv 0 \mod d \) and \( i < k_2 \) and otherwise, for \( n \equiv 1 \)

\[
y_n = \begin{cases} 
1, & \text{if } n \equiv m_1 \mod p \\
y_{n-k_2} - 1, & \text{if } n \equiv 0 \mod p \\
0, & \text{if } n \equiv m_2 \mod p
\end{cases}
\]

Note that

\[
2m_1 = 2m_1 + m_1 + m_2 = 3m_1 + m_2 = m_2 \mod p,
\]

and by (72), \( m_1 \neq 0 \mod p \).

As per usual, we will show that (81) holds for all \( n \geq 0 \). The result will then follow since \( \{y_i ; i \equiv 0 \mod p\} \) will be unbounded from below.

Suppose that \( y_n \) satisfies (81) for all \( n < N \). We will consider the three cases.

(i) \( (N = m_1 \mod p) \). In this case, \( N - m_1 = m_1 \mod p \), \( N - k_2 = m_1 - k_2 = m_1 \mod p \) and \( N - m_2 = 2m_1 = m_2 \mod p \). From the induction hypothesis and (29), we then have

\[
y_{N-k_2} - y_{N-m_2} = y_{N-m_1} = y_{N-m_1} - (y_{N-m_1} - 1) = 1.
\]

In addition, \( y_{N-k_2} - y_{N-m_2} = 1 - 0 = 1 \). Thus, \( y_N = 1 \), as required.

(ii) \( (N = 0 \mod p) \). In this case, \( N - k_1 = -k_1 = -m_1 = m_2 \mod p \) and \( N - m_1 = -m_1 = m_2 \mod p \), while, \( N - m_2 = m_2 = m_1 \mod p \), and thus \( y_{N-k_2} - y_{N-m_2} = y_{N-k_2} - 1 < y_{N-k_2} - y_{N-m_1} \). Thus, \( y_N = y_{N-k_2} - 1 \), as required.

(iii) \( (N = m_2 \mod p) \). Here, \( N - k_2 = m_2 \mod p \), \( N - m_2 = 0 \mod p \), \( N - k_1 = m_2 - k_1 = m_1 - m_2 = m_1 - m_2 = m_2 \mod p \) and \( N - m_1 = m_1 \mod p \). Thus, \( y_{N-k_1} - y_{N-m_1} = 0 < 2 \leq y_{N-k_2} - y_{N-m_2} \) and hence \( y_N = 0 \), as required.

The existence result is proven. There remains to prove that (71) holds for each of the unbounded solutions constructed above. Note that in each case,

\[
y_n = \begin{cases} 
y_{n-k_2} - 1, & \text{if } n \equiv 0 \mod p \\
y_{n-k_2}, & \text{if } n \equiv 0 \mod d \text{ and } n \notin 0 \mod p \\
y_{n-p}, & \text{otherwise}
\end{cases}
\]
In considering the limit supremum in (71), it thus suffices to consider the values \( \{y_{rp}\}/(rp) \}_{r \geq 0} \). Suppose that \( rp = tk_2 + q \), where \( q < k_2 \). Then,

\[
y_{rp} = y_{rp-k_2} - \delta_1 = y_{rp-2k_2} - (\delta_1 + \delta_2) = \cdots = y_{rp-\alpha_k} - (\delta_1 + \delta_2 + \cdots + \delta_i) = y_q - \sum_{j=1}^t \delta_j = y_q - 2 - \sum_{j=1}^t \delta_j
\]

(82)

where \( \delta_j = 1 \) if \( p \mid (rp-jk_2) \) and zero otherwise. Note that \( p \mid (rp-jk_2) \) if and only if \( (p/d) \mid j \), and hence, since \( t = \lfloor (rp)/k_2 \rfloor \), we have

\[
\sum_{j=1}^t \delta_j = \left\lfloor \frac{t}{p/d} \right\rfloor = \left\lfloor \frac{(rp)/k_2}{p/d} \right\rfloor
\]

(83)

and hence

\[
\limsup_{r \to \infty} \frac{|y_{rp}|}{rp} = \limsup_{r \to \infty} \frac{2 + \left\lfloor \frac{(rp)/k_2}{p/d} \right\rfloor}{rp} = \limsup_{r \to \infty} \frac{\left\lfloor (rp)/k_2 \right\rfloor}{rp} \frac{d}{p} = \frac{d}{p \kappa} = \frac{1}{\lcm(k_2,p)},
\]

(84)

We now turn to consideration of solutions to equation (4) where \( k_1 \neq m_1 + k_2 \).

4. The general \( L = 2 \) case in equation (1)

The following is a direct consequence of Theorems 8 and 9.

**Theorem 13.** Suppose that \( \gcd(r_1, s_1, r_2, s_2) = 1 \) for some \( r_1, s_1, r_2, s_2 \) satisfying \( r_1 = s_1 + r_2 \).

(a) If \( (k_1, m_1, k_2, m_2) = (r_1, s_1, r_2, s_2) \mod (s_1 + s_2) \), then there exists a non-trivial period \( s_1 + s_2 \) integer solution to equation (4).

(b) If \( (k_1, m_1, k_2, m_2) = (r_1, s_1, r_2, s_2) \mod r_2 \) and \( r_2 \neq s_2 \), then there exists a non-trivial period \( r_2 \mod s_2 \) integer solution to equation (4).

**Example.** Note that \( (r_1, s_1, r_2, s_2) = (2, 1, 1, 1) \) satisfies the assumption of Theorem 13(a), and hence all instances of equation (4) with \( k_1 \) even and \( m_1, k_2 \) and \( m_2 \) odd possess a non-trivial period two solution.

**Example.** Note that \( (r_1, s_1, r_2, s_2) = (4, 1, 3, 1) \) and \( (r_1, s_1, r_2, s_2) = (4, 1, 3, 2) \) satisfy the assumption of Theorem 13(b), and hence all instances of equation (4) with \( k_1, m_1 \equiv 1 \mod 3, k_2 \equiv 0 \mod 3 \) and \( m_2 \equiv 0 \mod 3 \) possess a non-trivial period three solution.

Several questions remain to be investigated regarding solutions to (4) even for small \( k_1, m_1, k_2 \) and \( m_2 \). We mention one particularly interesting case with the following conjecture.
Conjecture 2. Suppose that \( \gcd(k_1, m_1, k_2, m_2) = 1 \) and \( k_1 + k_2 = m_1 = m_2 = m \). Then, all integer solutions to (4) are strictly periodic, and moreover there exists a \( Q > 1 \) such that for all \( q > Q \) there exists some solution to (4) with prime period \( 2q \).

Numerical computations suggest that Conjecture 2 is quite plausible. For instance, when

\[
(k_1, m_1, k_2, m_2) = (1, 3, 2, 3),
\]

(85)

periodic solutions with prime period \( t \) have been shown to exist for all even \( 230 \leq t \leq 2290 \). While there appear to be periodic solutions with prime period \( t \) for almost all even \( t > 1 \), to this point, we have found only one admissible odd prime period – nine – in this case.

We note that under the assumptions in Conjecture 2, we have that

\[
y_n = \min\{y_{n-k_1} - y_{n-m}, y_{n-k_2} - y_{n-m}\} = \min\{y_{n-k_1}, y_{n-k_2}\} - y_{n-m},
\]

(86)

and \( m - \min\{k_1, k_2\} = \max\{k_1, k_2\} \). Hence

\[
y_{n-m} = \min\{y_{n-k_1}, y_{n-k_2}\} - y_{n-m},
\]

(87)

and \( \{y_n\} \) satisfies the interesting reversibility property that \( \{y_N, y_{N-1}, \ldots, y_0\} \) also satisfies (4) for all \( N \geq s \).

A particular consequence of (87) is that all eventually periodic solutions are, in this case, strictly periodic.

5. Concluding remarks

In this paper, we have focused on periodicity and boundedness for integer solutions to (1) with \( L \leq 2 \). It seems natural to consider more general composition-delay difference equations. In particular, suppose \( f \) and \( g \) are \( r \)-variable and \( t \)-variable functions respectively and \( K = [k_{i,j}] \) is an \( r \times t \) positive integer matrix. The \((f,g,K)\)-composition-delay equation is defined via

\[
y_n = f\left( g(y_{n-k_{1,1}}, \ldots, y_{n-k_{1,r}}), g(y_{n-k_{2,1}}, \ldots, y_{n-k_{2,r}}), \ldots, g(y_{n-k_{t,1}}, \ldots, y_{n-k_{t,r}}) \right),
\]

(88)

where \( k_{i,j} \geq 1 \) for \( 1 \leq i \leq r \) and \( 1 \leq j \leq t \), and \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in \mathbb{R} \), with

\[
s = \max\{k_{ij} : 1 \leq i \leq r \text{ and } 1 \leq j \leq t \}.
\]

(89)

We refer to \( f, g \) and \( K \) as the exterior function, the interior function and the delay matrix, respectively, for equation (88).

When \( f \) is the identity, (88) reduces to the standard higher-order difference equations which abound in the literature (see for instance [14] and the many other references on the topic).

In this paper, we have considered exclusively the case where \( f \) is the \( L \)-variable minimum function defined via

\[
f(x_1, x_2, \ldots, x_L) = \min\{x_1, x_2, \ldots, x_L\},
\]

(90)

and \( g \) is the difference function \( g(x,y) = x - y \).

A large number of interesting questions remain regarding solutions to equation (88), which are not handled in the present paper. In closing, we mention a few open problems.
Open Problem 1. Determine the behaviour of solutions to equation (1) for \( L > 2 \).

Open Problem 2. Determine the behaviour of solutions to equation (88) for the exterior 2-variable minimum function \( f \), defined via \( f(x, y) = \min\{x, y\} \), and \( r \)-variable linear interior functions \( g \) for \( r > 1 \).

Open Problem 3. Determine the behaviour of positive solutions to (88) for \( r \geq 2 \) for exterior minimum function, \( f \), as in (90) and interior function, \( g \), defined via \( g(x, y) = 1 + x/y \). Here, the case \( r = 1 \) is quite well understood (see for instance [3, 4, 14, 21, 24]).

It would be particularly interesting to have some general results regarding the behaviour of solutions to the \((f, g, K)\)-composition-delay equation in (88) in terms of analytic properties of \( f \) and \( g \) and algebraic properties of \( K \).

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Notes
1. Email: guyrt7@wfu.edu
2. Apparent unboundedness of particular solutions was verified by approximate computation.

References