Asymptotic behaviour of solutions to difference equations involving ratios of elementary symmetric polynomials

Kenneth S. Berenhaut* and Austin H. Jones

Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, USA
(Received 17 September 2010; final version received 22 October 2010)

This paper studies the behaviour of positive solutions of the recursive equation
\[ y_n = \left(\frac{e_{j,k}}{e_i}\right)(y_{n-1}, y_{n-2}, \ldots, y_{n-t_i}), \quad 0 \leq i, j \leq k, \]
where \( e_{m,k} \) is the \( m \)th elementary symmetric polynomial on \( k \) variables, \( t_i \geq 1 \) for \( 1 \leq i \leq k \), \( \gcd(t_1, t_2, \ldots, t_k) = 1 \) and \( y_{-1}, y_{-2}, \ldots, y_{-t_i} \in \mathbb{R}^+ \), with \( s = \max\{t_1, t_2, \ldots, t_k\} \). A variant of Newton’s inequalities is employed. Included among the results is a generalization of a particular case of Theorem 4.11 in E. A. Grove and G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall/CRC Press, Boca Raton, 2004.

Keywords: difference equations; asymptotic behaviour; symmetric functions; elementary symmetric polynomials; Newton’s inequalities; periodicity

AMS Subject Classification: 39A10; 39A11

1. Introduction

In this paper, we prove some results regarding the asymptotic behaviour of solutions to difference equations involving elementary symmetric polynomials. In particular, for fixed \( k \geq 0 \), consider the elementary symmetric polynomials, \( \{e_{j,k}\} \) in the \( k \) variables \( X_1, X_2, \ldots, X_k \), i.e., \( e_{0,k}(X_1, X_2, \ldots, X_k) = 1 \) and
\[ e_{j,k}(X_1, X_2, \ldots, X_k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} X_{i_1}X_{i_2}\cdots X_{i_j}, \quad 1 \leq j \leq k. \] (1)

When the inputs are clear from the context, we may denote \( e_{j,k}(X_1, X_2, \ldots, X_k) \) by \( e_j(X_1, X_2, \ldots, X_k) \) or simply \( e_j \) for \( 0 \leq j \leq k \). Note that every symmetric polynomial can be written as a polynomial in elementary symmetric polynomials (see for instance [14] or [15]). For some further discussion on the importance of symmetric polynomials, see for instance [12] and the extensive references in Ref. [7].

Rational difference equations involving elementary symmetric polynomials which have been considered recently include:
\[ x_n = \frac{1}{x_{n-t_1}} + \frac{1}{x_{n-t_2}} + \cdots + \frac{1}{x_{n-t_k}} \]
\[ = \frac{e_{k-1,k}(x_{n-t_1}, x_{n-t_2}, \ldots, x_{n-t_k})}{e_{k,k}(x_{n-t_1}, x_{n-t_2}, \ldots, x_{n-t_k})}, \] (2)

*Corresponding author. Email: berenhs@wfu.edu
\[ x_n = \frac{1 + x_{n-2}x_{n-3}}{x_{n-2} + x_{n-3}} \]

\[ = \frac{e_{0,2}(x_{n-1}, x_{n-2}) + e_{2,2}(x_{n-1}, x_{n-2})}{e_{1,2}(x_{n-1}, x_{n-2})}, \] (3)

and

\[ x_n = \frac{1 + x_{n-2}x_{n-3} + x_{n-1}x_{n-4} + x_{n-2}x_{n-5}}{x_{n-2} + x_{n-3} + x_{n-1} + x_{n-2}x_{n-4} + x_{n-3}x_{n-5}} \]

\[ = \frac{e_{0,3}(x_{n-1}, x_{n-2}, x_{n-3}) + e_{2,3}(x_{n-1}, x_{n-2}, x_{n-3})}{e_{1,3}(x_{n-1}, x_{n-2}, x_{n-3})}, \] (4)

for delays \( t_l \) satisfying \( t_l \geq 1 \) for \( 1 \leq l \leq k \) \((k = 2 \) and \( k = 3 \) in (3) and (4), respectively) and \( \gcd(t_1, t_2, \ldots, t_k) = 1 \).

All positive solutions to equations (3) and (4) were shown to converge in Refs [2,3], respectively. Solutions to (2) were shown to be asymptotically periodic with (not necessarily prime) period two in Ref. [9] (see also Section 4.5 in Refs [8,10] and Theorem 1.1, below).

For some further instances of difference equations involving elementary symmetric polynomials, see for instance [16].

The focus here will be on solutions to equations involving ratios of elementary symmetric polynomials, i.e.

\[ y_n = \left(\frac{e_{s,k}}{e_{s,k}}\right)(y_{n-t_1}, y_{n-t_2}, \ldots, y_{n-t_s}), \] (5)

where \( t_l \geq 1 \) for \( 1 \leq l \leq k \), \( \gcd(t_1, t_2, \ldots, t_k) = 1 \) and \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in \mathbb{R}^+ \), with

\[ s = \max\{t_1, t_2, \ldots, t_k\}. \] (6)

Note that if \( \{y_n\} \) satisfies (5), then \( \{x_n\} \) given by \( x_n = 1/y_n \) satisfies

\[ x_n = \left(\frac{e_{k-s,k}}{e_{k-s,k}}\right)(x_{n-t_1}, x_{n-t_2}, \ldots, x_{n-t_k}). \] (7)

A simple instance of (5) is

\[ y_n = \frac{e_{s,k}}{e_{s,k}} \frac{(y_{n-t_1}y_{n-t_2}) \cdots (y_{n-t_s})}{1}. \] (8)

In the case of (8), solutions clearly diverge if all initial values are larger than one and converge to zero if all initial values are less than one (see Theorem 1.4, below).

For the case of (5) where \( j = i + 1 \) and \( 0 \leq i \leq k - 1 \), i.e.

\[ y_n = \left(\frac{e_{j,k}}{e_{i+1,k}}\right)(y_{n-t_1}, y_{n-t_2}, \ldots, y_{n-t_k}), \] (9)

we will prove the following theorem regarding solutions. The case \( i = k - 1 \), i.e. (2) (and hence \( i = 0 \), as well, by (7)), was proven in Ref. [9] (see also [6,8,11] and Section 4.5 in [10]).
Theorem 1.1. Suppose \( \{y_n\} \) satisfies (9). If at least one \( i \) is even, then \( \{y_n\} \) converges to the unique equilibrium

\[
c = \frac{i+1}{\sqrt{k-i}} = \frac{\binom{k}{i}}{\binom{k}{i+1}}.
\]

Otherwise, \( \{y_n\} \) is asymptotically periodic with (not necessarily prime) period 2.

It is not difficult to verify that the equilibrium in (9) is indeed \( c = \sqrt{(i+1)/(k-i)} \), as in (10).

Instrumental in our proof of Theorem 1.1 is the following variant of Newton's Inequalities (see Theorem 2.1), which will be proven in Section 2.

Lemma 1.2. For \( X > 0 \) define the transformed value \( X^* \) via

\[
X^* = \max \left\{ \frac{X}{c^\prime}, c \right\},
\]

and suppose that \( X_1, X_2, \ldots, X_k > 0 \). Then,

\[
\left( \frac{e_i(X_1, X_2, \ldots, X_k)}{e_{i+1}(X_1, X_2, \ldots, X_k)} \right)^* \leq \frac{e_i(X_1^*, X_2^*, \ldots, X_k^*)}{k}.
\]

Note that it would have sufficed for our purposes to replace the inequality in (12) with the weaker part-metric inequality

\[
\left( \frac{e_i(X_1, X_2, \ldots, X_k)}{e_{i+1}(X_1, X_2, \ldots, X_k)} \right)^* \leq \max\{X_1^*, X_2^*, \ldots, X_k^*\},
\]

but the bound in (12) is stronger and somewhat more natural in the given context.

For completeness, we will also prove the following two theorems which follow fairly directly from Newton's Inequalities.

Theorem 1.3. Assume that \( \{y_n\} \) satisfies (5) with \( 1 \leq i \leq k \), and \( j = i - 1 \). Then,

(i) if \( i < \frac{k+1}{2} \) then \( \{y_n\} \) converges to infinity;
(ii) if \( i = \frac{k+1}{2} \) then \( \{y_n\} \) has a finite limit; and
(iii) if \( i > \frac{k+1}{2} \) then \( \{y_n\} \) converges to zero.

Theorem 1.4. For \( 0 \leq i < j + 1 \leq k \), there exist positive solutions \( \{y_n\} \) to (5) which converge to infinity and others which converge to zero.

For some further results regarding difference equations involving symmetric functions see for instance [1, 4, 16] and the references therein.

We close this section with the following open question.
OPEN QUESTION 1.5. Determine all \((i, j, k)\) and \((t_1, t_2, \ldots, t_k)\) such that all positive solutions to equation (5) converge to a unique equilibrium.

Note that computations suggest that there exist \((i, j, k)\) and \((t_1, t_2, \ldots, t_k)\) satisfying 1 \(\leq i \leq k - 2\) and \(i + 2 \leq j \leq k\) for which all solutions converge to the unique equilibrium

\[
c = \binom{k}{i} \binom{k}{j}^{1/(j-i+1)} = \frac{j(j-1) \cdots (i+1)}{(k-i)(k-(i+1)) \cdots (k-j+1)}^{1/(j-i+1)}, \tag{13}
\]

For instance, we have the following conjecture for the case \((i, j, k) = (2, 4, 4)\) and \((t_1, t_2, t_3, t_4) = (1, 4, 5, 6)\).

**Conjecture 1.6.** Suppose \(\{y_n\}\) is a positive solution to the equation

\[
y_n = \frac{1}{y_{n-1}y_{n-4}} + \frac{1}{y_{n-1}y_{n-6}} + \frac{1}{y_{n-4}y_{n-5}} + \frac{1}{y_{n-4}y_{n-6}} + \frac{1}{y_{n-5}y_{n-6}}. \tag{14}
\]

Then \(\{y_n\}\) converges to the unique equilibrium \(\sqrt{6}\).

For a simpler related conjecture, we have the following for \((i, j, k) = (1, 3, 3)\), \((t_1, t_2, t_3) = (a, b, b + a)\) and \(b > a > 1\).

**Conjecture 1.7.** Suppose \(b > a > 1\), \(\gcd(a, b) = 1\) and \(\{y_n\}\) is a positive solution to the equation

\[
y_n = \frac{1}{y_{n-a}y_{n-b}} + \frac{1}{y_{n-a}y_{n-(b+a)}} + \frac{1}{y_{n-b}y_{n-(b+a)}}. \tag{15}
\]

Then, \(\{y_n\}\) converges to the unique equilibrium \(\sqrt{3}\).

In Section 2, we provide proofs of Theorems 1.1, 1.3 and 1.4.

2. Main results

In this section, we include proofs of Theorems 1.1, 1.3 and 1.4. Essential to the proofs will be the following well-known set of inequalities (see for instance [13]).

**Theorem 2.1 (Newton's Inequalities).** For fixed \(k \geq 1\) and \(\{S_i\}_{0 \leq i \leq k}\) defined via

\[
S_i = S_i(x_1, x_2, \ldots, x_k) = \binom{e(x_1, x_2, \ldots, x_k)}{i}^k, \tag{16}
\]
we have for $X_1, X_2, \ldots, X_k > 0$ that

$$\frac{S_0}{S_1} \leq \frac{S_1}{S_2} \leq \cdots \leq \frac{S_{k-1}}{S_k}. \tag{17}$$

In addition, it follows that

$$\frac{\epsilon_0}{\epsilon_1} < \frac{\epsilon_1}{\epsilon_2} < \cdots < \frac{\epsilon_{k-1}}{\epsilon_k}. \tag{18}$$

Theorem 2.1 leads to the following lemma.

**Lemma 2.2.** For $0 \leq i < j \leq k$, the ratio $R_{ij}$ defined via

$$R_{ij}(X_1, X_2, \ldots, X_k) = \frac{\epsilon_{i,j}(X_1, \ldots, X_k)}{\epsilon_{i,j}(X_1, \ldots, X_k)} \tag{19}$$

is decreasing in each of its arguments.

**Proof.** The theorem clearly holds in the case $i = 0$ for $k \geq 1$. Otherwise, we have

$$R(X_1, X_2, \ldots, X_k) = \frac{X_k \epsilon_{i-1,k-1}(X_1, \ldots, X_{k-1}) + \epsilon_{i,k-1}(X_1, \ldots, X_{k-1})}{X_k \epsilon_{i-1,k-1}(X_1, \ldots, X_{k-1}) + \epsilon_{i,k-1}(X_1, \ldots, X_{k-1})} = \frac{X_k \epsilon_{i-1,k-1} + \epsilon_{i,k-1}}{X_k \epsilon_{i-1,k-1} + \epsilon_{i,k-1}}. \tag{20}$$

Hence

$$\frac{dR}{dX_k}(X_1, \ldots, X_k) = \frac{\epsilon_{i-1,k-1}(X_k \epsilon_{i-1,k-1} + \epsilon_{i,k-1}) - (X_k \epsilon_{i-1,k-1} + \epsilon_{i,k-1}) \epsilon_{i-1,k-1}}{(X_k \epsilon_{i-1,k-1} + \epsilon_{i,k-1})^2}$$

$$= \frac{\epsilon_{i-1,k-1} \epsilon_{i,k-1} - \epsilon_{i,k-1} \epsilon_{i-1,k-1}}{(X_k \epsilon_{i-1,k-1} + \epsilon_{i,k-1})^2} = \frac{(\epsilon_{i-1,k-1} / \epsilon_{i,k-1})^2 - (\epsilon_{i-1,k-1} / \epsilon_{i,k-1})}{(X_k \epsilon_{i-1,k-1} + \epsilon_{i,k-1})^2} < 0, \tag{21}$$

by (18), and since $R$ is symmetric in its arguments, the result follows. \qed

We are now in a position to prove Lemma 1.2.

**Proof.** (Proof of Lemma 1.2). We need to show that for fixed $r \geq 0$ and all positive $X_1, X_2, \ldots, X_k$ satisfying

$$X_1, X_2, \ldots, X_r \geq c \text{ and } X_{r+1}, X_{r+2}, \ldots, X_k \leq c, \tag{22}$$

$$Q_1 \equiv \frac{1}{k} e_1 \left( \frac{X_1}{c}, \ldots, \frac{X_r}{c}, \frac{X_{r+1}}{X_{r+1}}, \ldots, \frac{X_k}{c} \right) - \frac{1}{c} e_{i+1} \left( X_1, X_2, \ldots, X_k \right) \geq 0, \tag{23}$$
and

\[ Q_2 = \frac{1}{k} e_1 \left( \frac{X_1}{c}, \ldots, \frac{X_r}{c}, \frac{c}{X_{r+1}}, \ldots, \frac{c}{X_k} \right) - \frac{1}{c} e_i e_i^{e_{i+1}} \left( X_1, X_2, \ldots, X_k \right) \geq 0. \tag{24} \]

To prove (23), note that by Lemma 2.2,

\[ Q_1 \geq \frac{1}{k} e_1 \left( \frac{1}{c}, \ldots, \frac{1}{c}, \frac{c}{c}, \ldots, \frac{c}{c} \right) - \frac{1}{c} e_i e_i^{e_{i+1}} \left( c, \ldots, c, X_{r+1}, \ldots, X_k \right) \]

\[ = \frac{c}{k} e_1 \left( \frac{1}{c}, \ldots, \frac{1}{c}, \frac{1}{c}, \ldots, \frac{1}{c} \right) - \frac{1}{c} e_i e_i^{e_{i+1}} \left( c, \ldots, c, X_{r+1}, \ldots, X_k \right) \]

\[ = c \left( \frac{1}{e_k} e_k^{-1} \left( c, \ldots, c, X_{r+1}, \ldots, X_k \right) - \frac{1}{c} e_i e_i^{e_{i+1}} \left( c, \ldots, c, X_{r+1}, \ldots, X_k \right) \right) \]

\[ = c \left( \frac{S_{k-1}}{S_k} \left( c, \ldots, c, X_{r+1}, \ldots, X_k \right) - \frac{S_i}{S_{i+1}} \left( c, \ldots, c, X_{r+1}, \ldots, X_k \right) \right) \geq 0. \tag{25} \]

Similarly

\[ Q_2 \geq \frac{1}{k} e_1 \left( \frac{X_1}{c}, \ldots, \frac{X_r}{c}, 1, \ldots, 1 \right) - \frac{1}{c} e_i e_i^{e_{i+1}} \left( X_1, \ldots, X_r, c, \ldots, c \right) \]

\[ = \frac{1}{c} e_1 \left( \frac{X_1}{c}, \ldots, \frac{X_r}{c}, c, \ldots, c \right) - \frac{1}{c} e_i e_i^{e_{i+1}} \left( X_1, \ldots, X_r, c, \ldots, c \right) \]

\[ = \frac{1}{c} \left( \frac{S_i}{S_0} \left( X_1, \ldots, X_r, c, \ldots, c \right) - \frac{S_{i+1}}{S_i} \left( X_1, \ldots, X_r, c, \ldots, c \right) \right) \geq 0. \tag{26} \]

and the lemma is proven. \(\square\)

Now, for the purpose of proving Theorem 1.1, for \(\{y_i\}\) satisfying (9), consider the transformed sequence \(\{y_i^e\}\) defined via (11). In addition, set

\[ M_n = \max_{n - r \leq j \leq n} \{y^e_j\}, \quad n \geq 0. \tag{27} \]

Note that by Lemma 1.2, \(\{M_n\}\) is (not necessarily strictly) decreasing and, by definition, bounded below by one. Set

\[ \Omega = \lim_{n \to \infty} M_n. \tag{28} \]

We are now in a position to prove Theorem 1.1.

**Proof.** (Proof of Theorem 1.1) First, suppose \(V\) satisfies

\[ V + j = C_{j,l}t_1 + C_{j,2}t_2 + \cdots + C_{j,k}t_k, \tag{29} \]

for \(1 \leq j \leq s\), where \(C_{j,l} \geq 0\) for all \(j, l\) (that such a \(V\) exists follows, for instance, from Result A21 on p. 541 in Ref. [5], since \(gcd(t_1, t_2, \ldots, t_k) = 1\).

Now, fix \(\varepsilon > 0\). By the definition of \(\Omega\), it follows that \(y_n^e > \Omega - \varepsilon\) infinitely often.
Suppose, for some large $N > V + \varepsilon$ and $Q > 2N$,

$$y_Q^* > \frac{c}{\Omega - \varepsilon}, \quad \text{and} \quad y_n^* < \frac{c}{\Omega + \varepsilon} \text{ for } n > N. \quad (30)$$

This first inequality in (30) implies that either $y_Q < c/(\Omega - \varepsilon)$ or $y_Q > c(\Omega - \varepsilon)$. We will consider the latter case; the former case follows similarly.

In addition,

$$\frac{c}{\Omega + \varepsilon} < y_n < c(\Omega + \varepsilon) \text{ for } n > N. \quad (31)$$

Employing the assumptions on $Q$, $N$ and Lemma 2.2, we have that for all $1 \leq l \leq k$,

$$c(\Omega - \varepsilon) < y_Q < \frac{y_Q - i c_{k-1} x_{k-1} (\frac{c}{\Omega - \varepsilon}, \ldots, \frac{c}{\Omega - \varepsilon}) + c_{k-1} x_{k-1} (\frac{c}{\Omega - \varepsilon}, \ldots, \frac{c}{\Omega - \varepsilon})}{y_Q - i c_{k-1} x_{k-1} (\frac{c}{\Omega - \varepsilon}, \ldots, \frac{c}{\Omega - \varepsilon}) + c_{k+1} x_{k+1} (\frac{c}{\Omega - \varepsilon}, \ldots, \frac{c}{\Omega - \varepsilon})} \quad \text{for } l = 1$$

$$= \frac{y_Q - i c_{k-1} x_{k-1} (\frac{c}{\Omega - \varepsilon}) + c_{k+1} x_{k+1} (\frac{c}{\Omega - \varepsilon})}{y_Q - i c_{k-1} x_{k-1} (\frac{c}{\Omega - \varepsilon}) + c_{k+1} x_{k+1} (\frac{c}{\Omega - \varepsilon})} \quad \text{for } l > 1.$$

Rearranging then gives

$$y_Q - i c_{k-1} x_{k-1} (\frac{c}{\Omega - \varepsilon}) < \frac{c}{\Omega - \varepsilon} \text{ for } l = 1$$

$$= \frac{c}{\Omega - \varepsilon} \left( \frac{k-1}{i} - \frac{k+1}{i+1} \right) \quad \text{for } l > 1. \quad (32)$$

Similarly, the case $y_Q < c/(\Omega - \varepsilon)$ gives

$$y_Q - i c_{k-1} x_{k-1} (\frac{c}{\Omega - \varepsilon}) > c(\Omega - \varepsilon). \quad (33)$$

Now, suppose that at least one $t_j$ is even, and that the delays $t_1 = a$ and $t_J = b$ are odd and even, respectively (note that $\gcd(t_1, t_2, \ldots, t_k) = 1$ guarantees that such $I$ and $J$ exist).
Iteratively applying the argument employed in obtaining (32) and (33), we have
\[
\begin{align*}
y_{Q-a}, y_{Q-b}, y_{Q-3b}, y_{Q-5b}, \ldots, y_{Q-kb} &< \frac{c}{\Omega - \varepsilon}, \\
y_{Q-2a}, y_{Q-4a}, y_{Q-6a}, \ldots, y_{Q-2kb} &> c(\Omega - \varepsilon).
\end{align*}
\]
Letting \( \varepsilon \) tend to zero in (34) gives \( \Omega = 1 \), and the result follows in this case.

Now, suppose \( \eta_l, 1 \leq l \leq k \) are odd. Then, by (29), for \( 1 \leq j \leq s \),
\[
V + j = C_{j,1} \eta_1 + C_{j,2} \eta_2 + \cdots + C_{j,k} \eta_k = C_{j,1} + C_{j,2} + \cdots + C_{j,k} \mod 2.
\]
Note that for \( 1 \leq j \leq s, Q - V - j > 2N - V - j \geq N \). Hence, employing (31) and (35) and arguing similar to above, if \( V - j \) is odd, we have
\[
\frac{c}{\Omega + \varepsilon} < y_{Q-V-j} < \frac{c}{\Omega - \varepsilon},
\]
while, if \( V - j \) is even, we have
\[
c(\Omega - \varepsilon) < y_{Q-V-j} < c(\Omega + \varepsilon).
\]
Suppose \( V \) is even. Then, by Lemma 2.2 and (31),
\[
c(\Omega + \varepsilon) > y_{Q-V} > \frac{e_{i,k}}{e_{i+1,k}} \left( \frac{c}{\Omega - \varepsilon}, \ldots, \frac{c}{\Omega - \varepsilon} \right)
\]
\[
= \binom{i}{i} \left( \frac{c}{\Omega - \varepsilon} \right)^i
\]
\[
= \binom{i}{i+1} \left( \frac{c}{\Omega - \varepsilon} \right)^{i+1}
\]
\[
= \frac{i+1}{k-i} \left( \frac{\Omega - \varepsilon}{c} \right) = c(\Omega - \varepsilon).
\]
Similarly
\[
\frac{c}{\Omega + \varepsilon} < y_{Q-V+1} < \frac{c}{\Omega - \varepsilon}
\]
and inductively
\[
c(\Omega + \varepsilon) > y_{Q-V+j} > c(\Omega - \varepsilon),
\]
for \( j \geq 0 \) even, and
\[
\frac{c}{\Omega + \varepsilon} < y_{Q-V+j} < \frac{c}{\Omega - \varepsilon},
\]
for \( j \geq 1 \) odd. Letting \( \varepsilon \) tend to zero gives that \( \{y_j\} \) is asymptotically periodic with period 2. The case \( V \) odd is similar and the theorem is proven.

As mentioned earlier, the proofs of Theorems 1.3 and 1.4 are fairly straightforward.
Proof. (Proof of Theorem 1.3). By Lemma 2.2 and (5), for \( n \geq 0, \)
\[
\left( \begin{array}{c} i \\ j \end{array} \right) m^{i-j}_n \leq y_n \leq \left( \begin{array}{c} i \\ j \end{array} \right) M^{i-j}_n,
\]
(42)
where \( m_n = \min_{(i+j) \leq s \leq s_n} y_s \) and \( M_n = \max_{(i+j) \leq s \leq s_n} y_s. \) In particular, if \( j = i - 1, \) we have, with \( C = (k-i+1)/i \)
\[
Cm_n \leq y_n \leq CM_n.
\]
(43)
To prove the result in Case (i), note that for \( n \geq 0, \) \( y_n \geq Cm_n \) with \( C > 1, \) and hence \( \lim_{n \to \infty} m_n = \infty. \) In Case (iii), \( y_n \leq CM_n \) with \( C < 1, \) and hence \( \lim_{n \to \infty} M_n = 0. \) Finally, in Case (ii), the requirements of Theorem 1.9 in Ref. [10] are satisfied and hence every solution has a finite limit. \( \square \)

Proof. (Proof of Theorem 1.4). Set \( i = j \geq 2 \) and suppose \( \delta > 1. \) By (42), for \( n \geq 0, \)
\[
\left( \begin{array}{c} i \\ j \end{array} \right) m^{i-j}_n \leq y_n \leq \left( \begin{array}{c} i \\ j \end{array} \right) M^{i-j}_n,
\]
(44)
where \( m_n = \min_{(i+j) \leq s \leq s_n} y_s \) and \( M_n = \max_{(i+j) \leq s \leq s_n} y_s \) and \( C = \frac{\left( \begin{array}{c} i \\ j \end{array} \right)}{\left( \begin{array}{c} i \\ j \end{array} \right) M^{i-j}_n}. \) Hence, if \( y_i > (\delta/C)^{1/(\ell-1)} \) for \( -s \leq i \leq -1, \) then \( m_0 > (\delta/C)^{1/(\ell-1)} \) and
\[
y_0 \geq Cm^0_0 > Cm_0(\delta/C) = \delta m_0.
\]
(45)
Inductively, we then have that \( y_n \geq \delta m_0 \) for \( 0 \leq n \leq s - 1. \) Then,
\[
y_s \geq Cm^s_0 \geq C\delta^{-1}m^0_0 \geq \delta^s m_0.
\]
(46)
Continuing in this fashion, we have that \( \lim_{n \to \infty} y_n = \infty. \)
Similarly, if \( y_i < (1/(\delta C))^{1/(\ell-1)} \) for \( -s \leq i \leq -1, \) then \( M_0 < (1/(\delta C))^{1/(\ell-1)} \) and
\[
y_0 \leq CM^0_0 < CM_0 \left( \frac{1}{\delta C} \right) = M_0/\delta.
\]
(47)
Inductively, we then have that \( y_n \leq M_0/\delta \) for \( 0 \leq n \leq s - 1. \) Then,
\[
y_s \leq CM^s_0 \leq C\delta^{-1}M^0_0 \leq \delta^{-(\ell+1)}M_0 \leq M_0/\delta^\ell.
\]
(48)
Continuing in this fashion, we have that \( \lim_{n \to \infty} y_n = 0, \) and the theorem is proven. \( \square \)

References